11-27

$$\hat{I}_2 = \frac{1}{\sqrt{6}}(\hat{x} - 2\hat{y} + \hat{z})$$

← It can be shown.

$$\hat{I}_3 = \frac{1}{\sqrt{6}}(\hat{x}\ \hat{y} - 2\hat{z})?$$

This says the principal axes corresponding to I_1 lies along the body diagonal. The other two principal axes lie in a plane normal to the diagonal of the cube

Note that $\hat{I}_1 \cdot \hat{I}_2 = \hat{I}_1 \cdot \hat{I}_3 = 0$

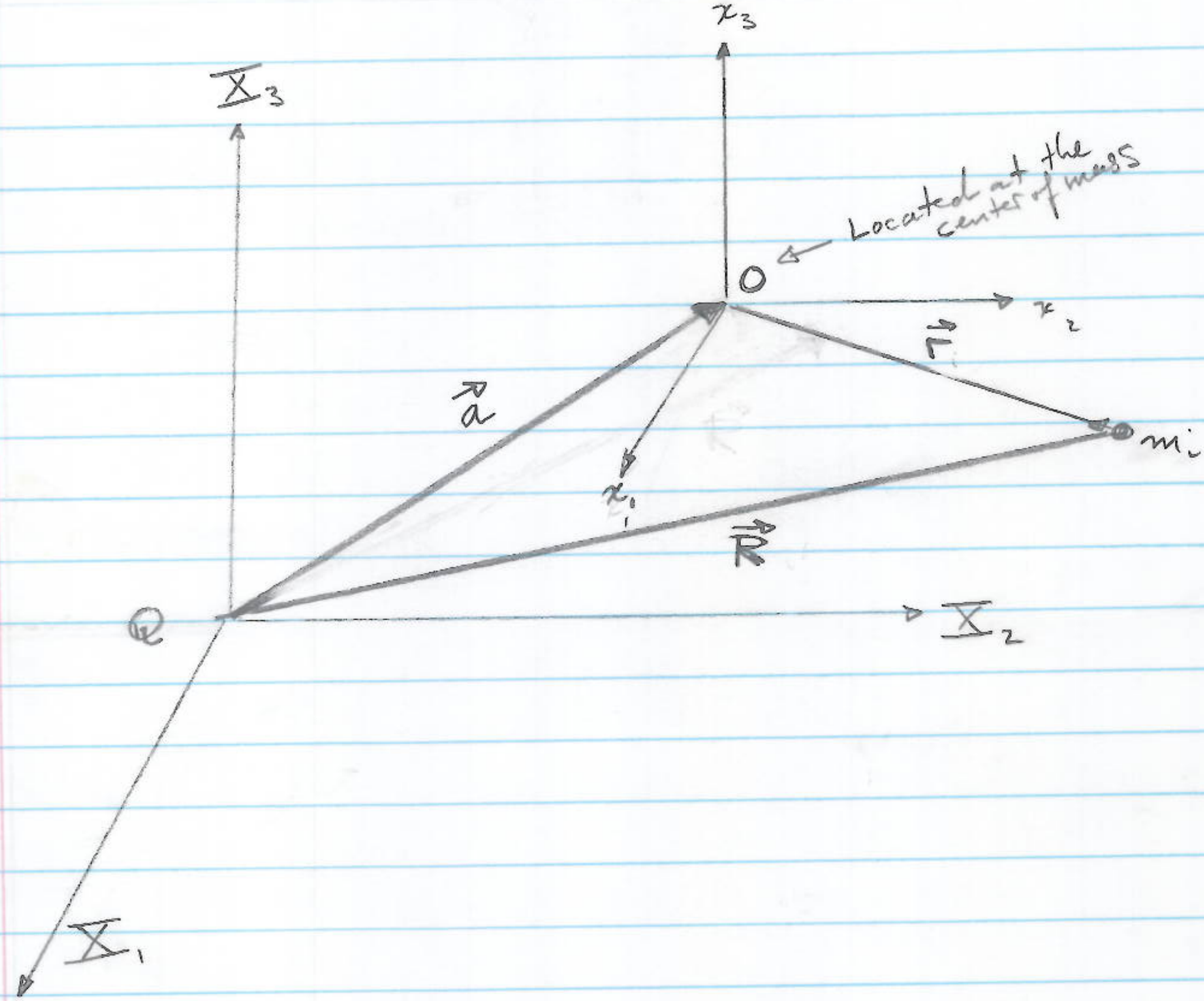
Moments of Inertia for Different Body Coordinate Systems.

For the kinetic energy to be separable into translational and rotational portions, it is, in general, necessary to choose a body coordinate system whose origin is the center of mass of the body

Now for certain geometrical shapes, it may not be convenient (or readily tractable) to compute the elements of the inertia tensor using a coordinate system with the origin @ the cm.

We shall consider another set of coordinate axes, in which the axes are parallel but shifted from the body coordinate axes going through the origin

11-28



The elements of the inertia tensor relative to the the X_i-axes can be written

$$J_{kl} = \sum_{i=1}^{n} m_i \left[\delta_{kl} R_i^2 - X_{ik} X_{il} \right]$$

If the vector connecting Q with O is $\vec{a}$, then the general vector $\vec{R}$ can be written as

$$\vec{R} = \vec{a} + \vec{r}$$

$$J_{kl} = \sum_{i=1}^{n} m_i \left[\left(\delta_{kl} \sum_{j=1}^{3} (x_{ij} + a_j)^2 \right) - (x_{ik} + a_k)(x_{il} + a_l) \right]$$

11-29

$$J_{kl} = \overbrace{\sum_{i=1}^{n} m_i \left[\delta_{kl} \sum_{j=1}^{3} x_{ij}^2 - x_{ik} x_{il} \right]}^{I_{kl}}$$

$$+ \sum_{i=1}^{n} m_i \left[\delta_{kl} \sum_{j=1}^{3} \left(2 x_{ij} a_j + a_j^2 \right) - \left(a_k x_{il} + a_l x_{ik} + a_k a_l \right) \right]$$

Identifying the first term as I_{kl} and rearranging

$$J_{kl} = I_{kl} + \sum_{i=1}^{n} m_i \left(\delta_{kl} \sum_{j=1}^{3} a_j^2 - a_k a_l \right)$$

$$+ \sum_{i=1}^{n} m_i \left(2 \delta_{kl} \sum_{j=1}^{3} x_{ij} a_j - a_k x_{ik} - a_l x_{il} \right)$$

Note that each term in the last summation involves a sum in the form of

$$\sum_{i=1}^{n} m_i x_{ij}$$

And because O is located at the center of mass

$$\sum_{i=1}^{n} m_i \vec{r}_i = 0$$

or for the j^{th} component (i.e x_1, x_2, x_3)

$$\sum_{i=1}^{n} m_i x_{ij} = 0$$

Therefore all such terms vanish and we have

$$J_{k\ell} = I_{k\ell} + \sum_{i=1}^{n} m_i \left(\delta_{k\ell} \sum_{j=1}^{3} a_j^2 - a_k a_\ell \right)$$

But
$$\sum_{i=1}^{n} m_i = \mathbb{M} \quad \text{and} \quad \sum_{j=1}^{3} a_j^2 \equiv a^2 \quad (\text{Pythagorus})$$

Solving for $I_{k\ell}$ gives us

$$\boxed{I_{k\ell} = J_{k\ell} - \mathbb{M}\left(a^2 \delta_{k\ell} - a_k a_\ell\right)}$$

We can now calculate the moment of inertia tensor with respect to the ORIGIN AT THE CENTER OF MASS once we know the moment of inertia tensor wrt origin Q.

This is the general form of

$\boxed{\text{Steiner's Parallel-Axis Theorem}}$

On pages 11-24 through 11-27 we calculated the inertia tensor (and principal axes) for the case of the origin (or pivot point) being at one corner of a cube of uniform density.

We now seek to determine the INERTIA TENSOR wrt the center of mass of the cube.

We found for the case of the pivot point being at a corner of the cube:

$$\{I\} = \begin{bmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{bmatrix} M b^2$$

The center of mass of the cube is at

$$x = x_1 = b/2$$
$$y = x_2 = b/2 \quad \leftarrow \text{See Fig. 11-6 p.422}$$
$$z = x_3 = b/2$$

The components of $\vec{a}$ are:

$$a_1 = a_2 = a_3 = b/2$$

We have

The diagonal components: $J_{11} = J_{22} = J_{33} = \frac{2}{3} M b^2$
The off-diagonal elements: $J_{12} = J_{13} = J_{23} = \frac{1}{4} M b^2$

From Steiner's Parallel-Axis Thm:

$$I_{k\ell} = J_{k\ell} - M(a^2 \delta_{k\ell} - a_k a_\ell)$$

We can calculate:

$$I_{11} = \frac{2}{3} M b^2 - M(a^2 - a_1^2)$$
$$= \frac{2}{3} M b^2 - M(a_2^2 + a_3^2) = \frac{2}{3} M b^2 - \frac{1}{2} M b^2$$

$$\boxed{I_{11} = I_{22} = I_{33} = \frac{1}{6} M b^2}$$

$$I_{12} = -\frac{1}{4} M b^2 + M(a_1 a_2) = -\frac{1}{4} M b^2 + M\left(\frac{b}{2} \frac{b}{2}\right)$$
$$= -\frac{1}{4} M b^2 + \frac{1}{4} M b^2$$

$$\boxed{I_{12} = I_{13} = I_{23} = 0}$$

The INERTIA TENSOR is therfore DIAGONAL

$$\{I\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\frac{1}{6} M b^2\right)$$

We see the principal axes are normal to the faces of the cube

$\{I\} = \frac{1}{6} M b^2 \{\mathbb{1}\}$, where $\{\mathbb{1}\}$ is the unit tensor.

Properties of the Inertia Tensor

The relation between the quantities $\vec{L}$ and $\vec{\omega}$ can be written as

$$\mathbb{I} = \frac{\vec{L}}{\vec{\omega}}$$

where $\mathbb{I}$ is the quotient of two vector quantities. In general, the quotient of two vector quantities is not necessarily a member of the same class as that of the two dividing factors. Hence we should <u>NOT</u> expect the ratio of two vectors to a vector. As a matter of fact, it is an altogether diferent quantity; it is

* A TENSOR of the SECOND RANK *

In Cartesian three-dimensional space, a Cartesian tensor $\mathbb{T}$ of the N^{th} rank may be defined as

(1) a quantity that has 3^N components $T_{ijk \ldots N}$
(2) Under orthogonal transformation of coordinates it obeys the following rule

$$T'_{ijk}(x') = a_{il}\, a_{jm}\, a_{kn} \cdots T_{lmn \ldots}(x)$$

where $a_{il}, a_{jm}, \ldots$ are the elements of transformation.

Since we shall not be using any other coordinates except cartesian, we shall simply use the term tensor $\mathbb{T}$ instead of Cartesian tensor $\mathbb{T}$.

For $\underline{N=0}$, i.e. a tensor of zero rank, we have $3^N = 3^0 = 1$.
That is, a tensor of zero rank has only one component; this quantity is invariant under an orthogonal transformation. $\Rightarrow$ We may say that a SCALAR IS a TENSOR OF ZERO RANK

For $\underline{N=1}$, i.e. a tensor of the first rank, we have $3^N = 3^1 = 3$.
That is, a tensor of the first rank will have three components. These components transform as

$$T_i' = a_{ij} T_j$$

which is similar to the transformation for a vector. Thus a vector is a tensor of the first rank and has three components

For $\underline{N=2}$, a tensor of second rank will have 9 components
$3^N = 3^2 = 9$
and will transform as

$$T_{ij}' = a_{il} a_{jm} T_{lm} .$$

This transformation is similar to a 3×3 square matrix with one fundamental difference – A matrix is not limited to only orthogonal transformations, which, by definition, a tensor of rank two is.

In spite of these differences, we shall make use of the properties of matrices in exploring the nature of tensors.

Another way of representing a tensor $\mathbb{T}$ is as a
DYADIC

Starting with the definition of angular momentum

$$\vec{L} = \sum_{i=1}^{n} m_i \left[r_i^2 \vec{\omega} - \vec{r}_i (\vec{r}_i \cdot \vec{\omega}) \right]$$

We may write this as

$$\vec{L} = \left(\sum_{i=1}^{n} m_i r_i^2 \right) \vec{\omega} - \underbrace{\left(\sum_{i=1}^{n} m_i \vec{r}_i \vec{r}_i \right)}_{\text{This is a dyad}} \cdot \vec{\omega}$$

We define a dyad as a simple pair of vectors written as $\vec{A}\vec{B}$. The quantity $\vec{A}\vec{B}$ has meaning only when it operates on other quantities. For example, we define the scalar dot product of a dyad with a vector $\vec{C}$ as

$$(\vec{A}\vec{B}) \cdot \vec{C} = \vec{A} (\vec{B} \cdot \vec{C}) \quad \leftarrow \text{This is a vector!}$$

or

$$\vec{C} \cdot (\vec{A}\vec{B}) = (\vec{C} \cdot \vec{A}) \vec{B}$$

The two vectors $\vec{A}(\vec{B} \cdot \vec{C})$ and $(\vec{C} \cdot \vec{A})\vec{B}$, in general, are NOT EQUAL.

DYAD SCALAR MULTIPLICATION IS NOT
~ COMMUTATIVE ~

nonstandard notation

if we let $\overleftrightarrow{T} = \vec{A}\vec{B}$
then we may write

$$\overleftrightarrow{T} \cdot \vec{C} = \vec{A}(\vec{B} \cdot \vec{C})$$
$$\vec{C} \cdot \overleftrightarrow{T} = (\vec{C} \cdot \vec{A})\vec{B}$$

Also

$$\overleftrightarrow{T} \cdot (\vec{C} \cdot \vec{D}) = \overleftrightarrow{T} \cdot \vec{C} + \overleftrightarrow{T} \cdot \vec{D}$$

$$\overleftrightarrow{T} \cdot c\vec{C} = c(\overleftrightarrow{T} \cdot \vec{C}), \text{ where } c = \text{const.}$$

A linear polynomial of dyads is called a

DYADIC

Such as $\vec{A}\vec{B} + \vec{C}\vec{D} + \cdots$

And any dyads may be expressed as a dyadic, if we express the vectors $\vec{A}$ and $\vec{B}$ in terms of unit vectors

$$\vec{C} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

$$\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{B} = b_1\hat{i} + b_2\hat{j} + c_3\hat{k}$$

The dyad $\vec{A}\vec{B}$ may then be written as a dyadic:

$$\overleftrightarrow{T} = \vec{A}\vec{B} = a_1b_1\hat{i}\hat{i} + a_1b_2\hat{i}\hat{j} + a_1b_3\hat{i}\hat{k} + a_2b_1\hat{j}\hat{i} + a_2b_2\hat{j}\hat{j} + a_2b_3\hat{j}\hat{k} + a_3b_1\hat{k}\hat{i} + a_3b_2\hat{k}\hat{j} + a_3b_3\hat{k}\hat{k}$$

Thus in matrix notation, we may write

$$\overleftrightarrow{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix}$$

We see that any given component of $\overleftrightarrow{T}$ may be written as T_{ij}

In component form, we may write

$$\vec{C} = \sum_{j=1}^{3} c_j \hat{u}_j$$

Here,

$$\hat{u}_j = \begin{cases} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{cases} \text{ are the unit vectors}$$

We have therefore

$$(\overleftrightarrow{T} \cdot \vec{C})_i = \sum_{j=1}^{3} T_{ij} C_j$$

$$(\vec{C} \cdot \overleftrightarrow{T})_i = \sum_{j=1}^{3} C_j T_{ji}$$

$$T_{ij} = \hat{u}_i \cdot (\overleftrightarrow{T} \cdot \hat{u}_j) = (\hat{u}_i \cdot \overleftrightarrow{T}) \cdot \hat{u}_j$$

Here $\overleftrightarrow{T} = \sum_{i,j=1}^{3} T_{ij} \hat{u}_i \hat{u}_j$

We define the unit dyadic $\overleftrightarrow{\mathbb{1}} = \hat{i}\hat{i} + \hat{j}\hat{j} + \hat{k}\hat{k}$ and $\overleftrightarrow{\mathbb{1}}$ behaves exactly like the unit matrix

$$\overleftrightarrow{\mathbb{1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overleftrightarrow{\mathbb{1}} \cdot \vec{A} = \vec{A} \cdot \overleftrightarrow{\mathbb{1}} = \vec{A}$$

We see that a tensor of second rank is very similar to a 3x3 matrix in its representation.

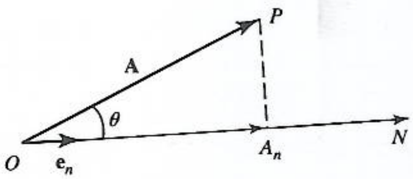


Figure 5.14 Component A_n of vector **A** along the N-axis.

The result of Eq. (5.63) may be written in a general form. Suppose we want to find component A_n of **A** along an arbitrary axis N that has a unit vector $\mathbf{e}_n$ along this axis, as shown in Fig. 5.14. We may write

$$A_n = \mathbf{A} \cdot \mathbf{e}_n \tag{5.64}$$

5.6 DIRECTIONAL COSINES

Let us start with vector **A** expressed in the form of Eq. (5.50) written in a slightly different form as

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}} = A\left(\frac{A_x}{A}\hat{\mathbf{i}} + \frac{A_y}{A}\hat{\mathbf{j}} + \frac{A_z}{A}\hat{\mathbf{k}}\right) = A\mathbf{e}_\mathbf{A} \tag{5.65}$$

where $\mathbf{e}_\mathbf{A}$ is a unit vector in the direction of **A**. A_x/A is equal to the cosine of the angle between A and the X-axis. Thus

$$\frac{A_x}{A} = \cos(\mathbf{A}, \mathbf{X}) = \cos(\mathbf{A}, \hat{\mathbf{i}}) = \alpha \tag{5.66}$$

$$\frac{A_y}{A} = \cos(\mathbf{A}, \mathbf{Y}) = \cos(\mathbf{A}, \hat{\mathbf{j}}) = \beta \tag{5.67}$$

$$\frac{A_y}{A} = \cos(\mathbf{A}, \mathbf{Z}) = \cos(\mathbf{A}, \hat{\mathbf{k}}) = \gamma \tag{5.68}$$

where α, β, and γ are called the *directional cosines* of the line representing **A**. Thus Eq. (5.65) may be written as

$$\mathbf{A} = A(\alpha\hat{\mathbf{i}} + \beta\hat{\mathbf{j}} + \gamma\hat{\mathbf{k}}) = A\mathbf{e}_\mathbf{A} \tag{5.69}$$

That is,

$$\mathbf{e}_\mathbf{A} = \alpha\hat{\mathbf{i}} + \beta\hat{\mathbf{j}} + \gamma\hat{\mathbf{k}} \tag{5.70}$$

which expresses the unit vector $\mathbf{e}_\mathbf{A}$ along **A** in terms of the directional cosines of **A** and the unit vectors. From Eq. (5.65), we may also write

$$\mathbf{A} \cdot \mathbf{A} = A^2\left[\left(\frac{A_x}{A}\right)^2 + \left(\frac{A_y}{A}\right)^2 + \left(\frac{A_z}{A}\right)^2\right] \tag{5.71}$$

By definition, $\mathbf{A} \cdot \mathbf{A} = A^2$; hence Eq. (5.71) yields

$$\left[\left(\frac{A_x}{A}\right)^2 + \left(\frac{A_y}{A}\right)^2 + \left(\frac{A_z}{A}\right)^2\right] = 1 \tag{5.72}$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \tag{5.73}$$

That is, the sum of the squares of the directional cosines of any line is equal to 1.

$$\Delta y := 8 \cdot \mathrm{cm} \qquad \Delta s := 64 \cdot \mathrm{cm}^2$$

$$\frac{\mathrm{vix} - \mathrm{vfx}}{\Delta y} = 2 \cdot \mathrm{sec}^{-1} \qquad \frac{\mathrm{vix} - \mathrm{vfx}}{\Delta y} \cdot \Delta s = 128 \cdot \mathrm{cm}^2 \cdot \mathrm{sec}^{-1}$$

5.9 COORDINATE TRANSFORMATIONS

The results of the application of any physical law to a given system must be independent of the coordinate system and the location of the origin of the coordinate system as well. Vectors have this special feature and hence are frequently used in various situations. Thus it becomes relevant to know the procedure by which vectors transfer from one coordinate system to another, that is, to investigate the properties of such transformations. Futhermore, it is convenient and useful to describe these vectors as well as their transformations in matrix notation.

We start with the description of a scalar in different coordinate systems. Suppose mass M is placed at point P and its coordinates are (x, y) in the XY system and (x', y') in the $X'Y'$ system, as shown in Fig. 5.22. The coordinates of the mass are different in the two coordinate systems, but the mass remains constant; that is,

$$M(x, y) = M(x', y') = \text{constant} \tag{5.133}$$

Quantities that are invariant under a coordinate transformation are called scalars.

Let us now investigate the procedure for coordinate transformation of vectors. Consider a point P that has coordinates (x_1, x_2, x_3) with respect to the $X_1X_2X_3$ coordinate system and (x_1', x_2', x_3') with respect to the $X_1'X_2'X_3'$ coordinate system, as shown in Fig. 5.23. Note that, for convenience, we are using x_1, x_2, x_3 instead of x, y, z.

For simplicity's sake, let us find the relationship between (x_1', x_2') and (x_1, x_2). Referring to Fig. 5.24, x_1' is given by

$$\begin{aligned} x_1' &= Oa = Ob + bc + ca \\ &= x_1 \cos\theta + ce \sin\theta + cP \sin\theta \\ &= x_1 \cos\theta + (ce + cP) \sin\theta \end{aligned}$$

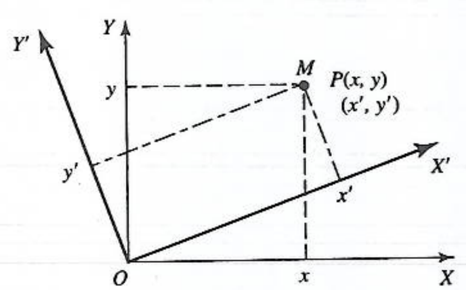


Figure 5.22 Coordinates of a scalar mass M at point P.

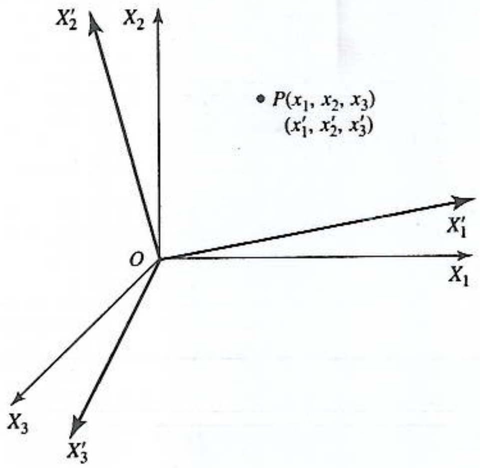


Figure 5.23 Coordinates of point P in two different coordinate systems rotated with respect to each other are (x_1, x_2, x_3) and (x_1', x_2', x_3').

$$= x_1 \cos\theta + eP \sin\theta$$

$$\boxed{x_1' = x_1 \cos\theta + x_2 \sin\theta}$$

or
$$x_1' = x_1 \cos\theta + x_2 \cos\left(\frac{\pi}{2} - \theta\right) \tag{5.134}$$

Similarly,
$$\boxed{x_2' = -x_1 \sin\theta + x_2 \cos\theta}$$

or
$$x_2' = x_1 \cos\left(\frac{\pi}{2} + \theta\right) + x_2 \cos\theta \tag{5.135}$$

a rotation about the x_3 axis

$$\underset{\sim}{\lambda} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{x}' = \underset{\sim}{\lambda} \vec{x}$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

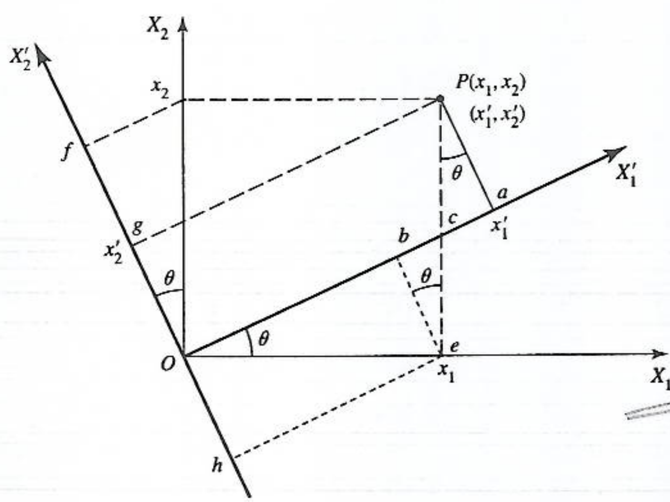


Figure 5.24 To calculate (x_1', x_2') in terms of (x_1, x_2).

Thus we have been able to express x_1' and x_2' in terms of x_1, x_2 and the cosines of angle θ. The notation can be simplified by using directional cosines. Let α_1 be the cosine of the angle between the X_1'-axis and X_1-axis or between the unit vectors $\hat{\mathbf{x}}_1'$ and $\hat{\mathbf{x}}_1$; that is,

$$\alpha_1 \equiv \cos(X_1', X_1) = \cos(\hat{\mathbf{x}}_1', \hat{\mathbf{x}}_1) = \hat{\mathbf{x}}_1' \cdot \hat{\mathbf{x}}_1 = \cos\theta$$

Similarly,

$$\begin{aligned}
\alpha_2 &\equiv \cos(X_1', X_2) = \cos(\hat{\mathbf{x}}_2', \hat{\mathbf{x}}_2) = \hat{\mathbf{x}}_2' \cdot \hat{\mathbf{x}}_2 = \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \\
\beta_1 &\equiv \cos(X_2', X_1) = \cos(\hat{\mathbf{x}}_2', \hat{\mathbf{x}}_1) = \hat{\mathbf{x}}_2' \cdot \hat{\mathbf{x}}_1 = \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta \\
\beta_2 &\equiv \cos(X_2', X_2) = \cos(\hat{\mathbf{x}}_2', \hat{\mathbf{x}}_2) = \hat{\mathbf{x}}_2' \cdot \hat{\mathbf{x}}_2 = \cos\theta \qquad (5.136)
\end{aligned}$$

Thus Eqs. (5.134) and (5.135) may be written as

$$x_1' = \alpha_1 x_1 + \alpha_{2x2} \qquad (5.137)$$

$$x_2' = \beta_1 x_1 + \beta_2 x_2 \qquad (5.138)$$

We may extend these equations to a three-dimensional case.

If we deal with a point P in space with coordinates (x_1, x_2, x_3) in the $X_1X_2X_3$ system and (x_1', x_2', x_3') in the $X_1'X_2'X_3'$ system, then

$$x_1' = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \qquad (5.139)$$

$$x_2' = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 \qquad (5.140)$$

$$x_3' = \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3 \qquad (5.141)$$

where γ_1 is the cosine of the angle between $\hat{\mathbf{x}}_3'$ and $\hat{\mathbf{x}}_1$. γ_2, γ_3, α_3, and β_3 have similar meanings. The reverse transformation, that is, x_1, x_2, x_3 in terms of x_1', x_2', x_3', may be written as

$$x_1 = \alpha_1 x_1' + \beta_1 x_2' + \gamma_1 x_3' \qquad (5.142)$$

$$x_2 = \alpha_2 x_1' + \beta_2 x_2' + \gamma_2 x_3' \qquad (5.143)$$

$$x_3 = \alpha_3 x_1' + \beta_3 x_2' + \gamma_3 x_3' \qquad (5.144)$$

where α_1 is the cosine of the angle between the X_1-axis and X_1'-axis, and the other directional cosines have similar meaning.

The transformation equations, Eqs. (5.142) to (5.144), may be written in a much neater and more compact form by using the following notation. Let λ_{ij} be the cosine of the angle between the X_i'-axis and X_j-axis; that is, the directional cosine λ_{ij} is

$$\boxed{\lambda_{ij} \equiv \cos(X_i', X_j) = \hat{\mathbf{x}}_i' \cdot \hat{\mathbf{x}}_j} \qquad (5.145)$$

and
$$\lambda_{ji} \equiv \cos(X_j, X_1') = \hat{\mathbf{x}}_j \cdot \hat{\mathbf{x}}_i' \tag{5.146}$$

The coordinates x_1', x_2', x_3' may be expressed in terms of x_1, x_2, x_3 as

$$x_1' = \lambda_{11}x_1 + \lambda_{12}x_2 + \lambda_{13}x_3 \tag{5.147}$$

$$x_2' = \lambda_{21}x_1 + \lambda_{22}x_2 + \lambda_{23}x_3 \tag{5.148}$$

$$x_3' = \lambda_{31}x_1 + \lambda_{32}x_2 + \lambda_{33}x_3 \tag{5.149}$$

while the reverse transformation is

$$x_1 = \lambda_{11}x_1' + \lambda_{21}x_2' + \lambda_{31}x_3' \tag{5.150}$$

$$x_2 = \lambda_{12}x_1' + \lambda_{22}x_2' + \lambda_{32}x_3' \tag{5.151}$$

$$x_3 = \lambda_{13}x_1' + \lambda_{23}x_2' + \lambda_{33}x_3' \tag{5.152}$$

Using summation notation, these transformations may be written as

$$x_i' = \sum_{j=1}^{3} \lambda_{ij} x_j, \qquad i = 1, 2, 3 \tag{5.153}$$

$$x_i = \sum_{j=1}^{3} \lambda_{ji} x_j', \qquad i = 1, 2, 3 \tag{5.154}$$

where λ_{ij} may be thought of as elements of a 3 by 3 square matrix $\boldsymbol{\lambda}$ defined as

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \tag{5.155}$$

The matrix $\boldsymbol{\lambda}$ is called a *transformation matrix* or a *rotation matrix* and determines the properties of the coordinates of a point under transformation.

According to Eq. (5.155), we need nine quantities λ_{ij} to cause the coordinate transformation of a point. But looking further into the properties of $\boldsymbol{\lambda}$, we find that not all the quantities λ_{ij} are independent. To understand this, we look at two geometrical relations. In Fig. 5.25, the line OP makes angles θ_1, θ_2, and θ_3 with the X_1-, X_2-, and X_3-axes, respectively. Hence directional cosines of the straight line OP are $\cos\theta_1$, $\cos\theta_2$, and $\cos\theta_3$. As discussed in Section 5.6, the sum of the squares of the direction cosines of any line is equal to unity; that is (after replacing α, β, and γ by θ_1, θ_2, and θ_3, respectively)

$$\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 = 1 \tag{5.156}$$

Now with reference to Fig. 5.26, if a line OP makes angles θ_1, θ_2, θ_3 and line OQ makes angles θ_1', θ_2', θ_3' with the axes $X_1X_2X_3$, the cosine of the angle between these lines is given by

$$\cos\theta = \cos\theta_1 \cos\theta_1' + \cos\theta_2 \cos\theta_2' + \cos\theta_3 \cos\theta_3' \tag{5.157}$$

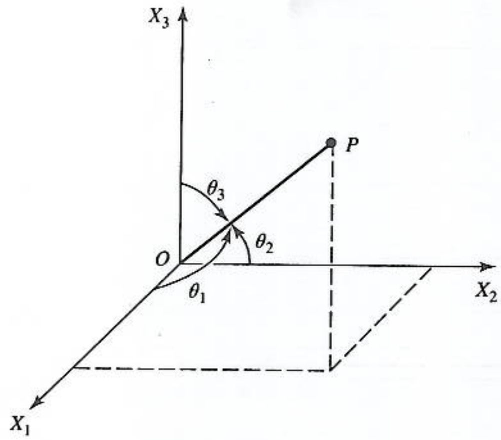


Figure 5.25 The angles θ_1, θ_2, θ_3 that OP makes with the three axes are used for calculating directional cosines.

Let us now consider a set of axes $X_1X_2X_3$. Each of these, when rotated through an angle θ, results in a new set of axes $X_1'X_2'X_3'$. Let us describe the X_1'-axis in the $X_1X_2X_3$ system. Its direction cosines are λ_{11}, λ_{12}, λ_{13}, while for the X_2'-axis in the $X_1X_2X_3$. system, they are λ_{21}, λ_{22}, λ_{23}. Since X_1' is perpendicular to X_2', the angle θ is $\pi 2$; when we apply Eq. (5.157), we get

$$\lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} + \lambda_{13}\lambda_{23} = \cos\theta = \cos\frac{\pi}{2} = 0 \tag{5.158}$$

which may be written as

$$\sum_{k=1}^{3} \lambda_{1k}\lambda_{2k} = 0 \tag{5.159}$$

In general, applying this to the other two combinations, we get

$$\sum_{k=1}^{3} \lambda_{ik}\lambda_{jk} = 0, \qquad \text{if } i \neq j \tag{5.160}$$

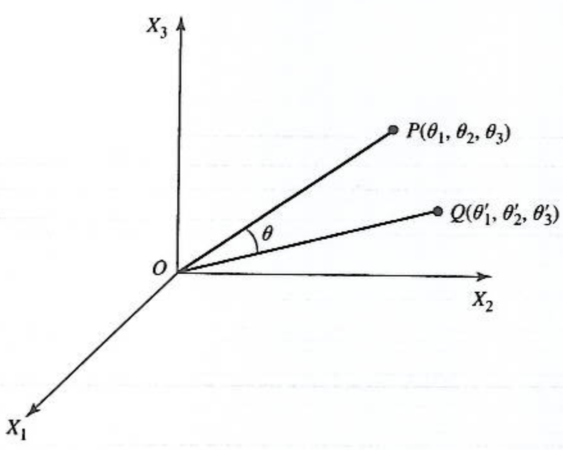


Figure 5.26 For calculating θ between two lines OP and OQ.

Similarly, if we apply Eq. (5.156) to the three axes X_1', X_2', X_3' separately described in the $X_1X_2X_3$ system, we get

$$\begin{aligned} \lambda_{11}^2 + \lambda_{12}^2 + \lambda_{13}^2 &= 1 \\ \lambda_{21}^2 + \lambda_{22}^2 + \lambda_{23}^2 &= 1 \\ \lambda_{31}^2 + \lambda_{32}^2 + \lambda_{33}^2 &= 1 \end{aligned} \tag{5.161}$$

which may be written in compact form as

$$\sum_{k=1}^{3} \lambda_{ik}\lambda_{jk} = 1, \quad \text{if } i = j \tag{5.162}$$

Equations (5.160) and (5.162) are called *orthogonality conditions* and apply to any set of coordinate systems in which the coordinate axes are mutually perpendicular; that is, the *systems* are *orthogonal*. Equations (5.160) and (5.162) may be combined into one as

$$\boxed{\sum_{k=1}^{3} \lambda_{jk}\lambda_{jk} = \delta_{ij}} \tag{5.163}$$

where δ_{ij} is called the *Kronecker* delta, which has the properties

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \tag{5.164}$$

Thus Eq. (5.163) results in six relations between the directional cosines, thereby reducing the number of independent quantities λ_{ij} in the matrix $\boldsymbol{\lambda}$ to only three.

The transformation matrix $\boldsymbol{\lambda}$ described here can be used to describe two different but closely related transformation:

1. *Coordinate transformation:* In this case, point P is fixed, while the base vectors are transformed (say from X_1 X_2 to $X_1'X_2'$), causing the coordinates of point P to change, as shown in Fig. 5.27(a). This is the interpretation we have explained here.
2. *Point transformation:* The alternative is to keep the coordinates (or base vectors) fixed and let point P rotate to point P', as shown in Fig. 5.27(b), always keeping the distance from the origin constant.

Let us reconsider Fig. 5.27(a) and (b). In Fig. 5.27(a), axes X_1 and X_2 are fixed and are the reference axes, while axes X_1' and X_2' are obtained by a rotation through an angle θ. Thus the coordinates of point P (x_1', x_2') in the rotated coordinate system are given in terms of the coordinates (x_1, x_2) in the fixed coordinate system as

$$x_1' = x_1 \cos\theta + x_2 \sin\theta \tag{5.165a}$$

$$x_2' = -x_1 \sin\theta + x_2 \cos\theta \tag{5.165b}$$

In this case the transformation acts on the axes and is called a *coordinate transformation.* The same result can be obtained if we keep the axes fixed but rotate point P through an angle θ (in

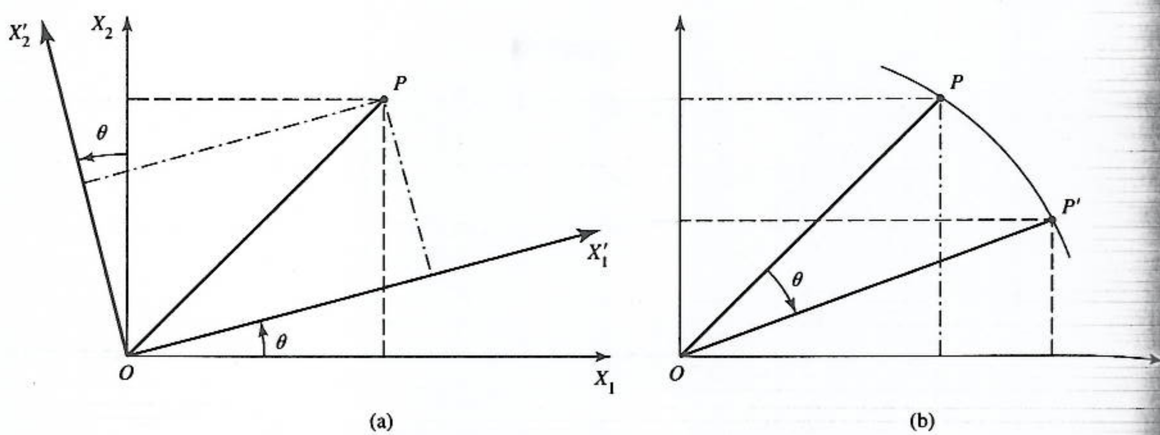


Figure 5.27 (a) Coordinate transformation and (b) point transformation.

the direction opposite to that in which the axes were rotated) to P'. The coordinates of P' are again given by Eqs. (5.165). This type of transformation, which acts on a point, is called a *point transformation.* The two types of transformations are completely equivalent.

Finally, a set of quantities $A_i(A_1, A_2, A_3)$ in an unprimed system may be transformed to a primed system by means of a transformation matrix $\boldsymbol{\lambda}$, resulting in [see Eqs. (5.153)]

$$A_i' = \sum_j \lambda_{ij} A_j \tag{5.166}$$

The quantities that obey such transformation rules are called vectors; that is, $A_i(A_1, A_2, A_3) \equiv \mathbf{A}$ is a vector quantity.

We have considered a transformation matrix $\boldsymbol{\lambda}$ that is a 3 by 3 square matrix; that is, the number of rows is equal to the number of columns. The matrix may not always be a square. For example, the coordinates (x_1, x_2, x_3) of a point may be represented by a *column matrix* x as

$$x \equiv \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{5.167}$$

or a *row matrix*

$$x \equiv (x_1 \;\; x_2 \;\; x_3) \tag{5.168}$$

A common practice is to use the column matrix given by Eq. (5.167) for representing a vector, and we shall use this convention. Thus the coordinates $x_i(x_1, x_2, x_3)$ and $x_i'(x_1', x_2', x_3')$ of point P with respect to the two reference coordinates, the $X_1X_2X_3$ and X_1', X_2', X_3' systems, respectively, may be expressed in matrix representation. Thus the transformation equations given by Eq. (5.153)

$$x_i' = \sum_{j=1}^{3} \lambda_{ij} x_j, \qquad i = 1, 2, 3 \tag{5.153}$$

may be written in matrix notation as

$$x' = \lambda x \tag{5.169}$$

This is equivalent to

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{5.170}$$

which is the same thing as

$$\begin{aligned} x_1' &= \lambda_{11}x_1 + \lambda_{12}x_2 + \lambda_{13}x_3 \\ x_2' &= \lambda_{21}x_1 + \lambda_{22}x_2 + \lambda_{23}x_3 \\ x_3' &= \lambda_{31}x_1 + \lambda_{32}x_2 + \lambda_{33}x_3 \end{aligned} \tag{5.171}$$

Note that the multiplication in Eqs. (5.170) or (5.171) is possible only if (1) x and x' are column matrices, and (2) the number of columns in $\boldsymbol{\lambda}$ must be equal to the number of rows in x. In general, if we want to multiply matrix $\boldsymbol{A}$ with matrix $\boldsymbol{B}$, the resulting matrix $\boldsymbol{C}$ is given by

$$C = AB \tag{5.172}$$

where the number of columns in matrix A must be equal to the number of rows in $\boldsymbol{B}$. Any element C_{ij} of matrix $\boldsymbol{C}$ is given by

$$C_{ij} = [AB]_{ij} = \sum_k A_{ik}B_{kj} \tag{5.173}$$

In general, matrix multiplication is not commutative; that is,

$$AB \neq BA \tag{5.174}$$

PROBLEMS

5.1. Prove the following inequalities:
(a) $|\mathbf{A} + \mathbf{B}| \leqslant |\mathbf{A}| + |\mathbf{B}|$ (b) $|\mathbf{A} \cdot \mathbf{B}| \leqslant |\mathbf{A}||\mathbf{B}|$ (c) $|\mathbf{A} \times \mathbf{B}| \leqslant |\mathbf{A}||\mathbf{B}|$

5.2 Find the resultant of three forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ in terms of their magnitudes F_1, F_2, and F_3 and angles θ_1, θ_2, and θ_3 between each pair of forces. Also find an expression for the angle α between the resultant force $\mathbf{F}$ and the component force $\mathbf{F}_1$.

5.3. Given the vectors $\mathbf{A} = (4, -2, 6)$ and $\mathbf{B} = (1, 3, -4)$, calculate (a) $\mathbf{A} + \mathbf{B}$, (b) $\mathbf{A} - \mathbf{B}$, (c) $3\mathbf{A} + 2\mathbf{B}$, (d) $3\mathbf{A} - 2\mathbf{B}$, (e) $A, B, A^2 + B^2$, (f) $\mathbf{A} \cdot \mathbf{B}$, (g) the angle between $\mathbf{A}$ and $\mathbf{B}$, (h) $\mathbf{A} \times \mathbf{B}$, (i) the component of $\mathbf{B}$ in the direction of $\mathbf{A}$, and (j) the directional cosines of $\mathbf{A}$ and $\mathbf{B}$.

5.4. Given two vectors $\mathbf{A} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ and $\mathbf{B} = -2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$, calculate (a) $\mathbf{A} + \mathbf{B}$, (b) $\mathbf{A} - \mathbf{B}$, (c) $3\mathbf{A} + 2\mathbf{B}$, (d) $3\mathbf{A} - 2\mathbf{B}$, (e) $A, B, A^2 + B^2$, (f) $\mathbf{A} \cdot \mathbf{B}$, (g) the angle between $\mathbf{A}$ and $\mathbf{B}$, (h) $\mathbf{A} \times \mathbf{B}$, (i) the component of $\mathbf{B}$ in the direction of $\mathbf{A}$, and (j) the directional cosines of $\mathbf{A}$ and $\mathbf{B}$.

5.5. Find the cosine of the angle between vectors $\mathbf{A} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\mathbf{B} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$.

5.6. Prove that the diagonals of an equilateral parallelogram are perpendicular.

5.7. Find a unit vector $\hat{\mathbf{n}}$ that is perpendicular to vectors $\mathbf{A} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $\mathbf{B} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$.

5.8. $\mathbf{A} = 2\hat{\mathbf{i}} + c\hat{\mathbf{j}} + \hat{\mathbf{k}}$ is perpendicular to $\mathbf{B} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$. What is the value of c?

The transformational properties in orthogonal cartesian coordinates may be used here.

Let us start with a vector $\vec{L}$, the angular momentum, fixed in an inertial reference frame.

$$(*) \qquad L_k = \sum_l I_{kl}\,\omega_l$$

In a body coordinate system that is simply rotated with respect to the space coordinates, the angular momentum $\vec{L}'$ must have an analogous form

$$(1) \qquad L'_i = \sum_j I'_{ij}\,\omega'_j$$

Using the transformation properties of vectors, we may write the transformation of $\vec{L}$ and $\vec{\omega}$ as [see handout on coordinate transformations]

$$x_i = \sum_j \lambda^{\dagger}_{ij}\,x'_j = \sum_j \lambda_{ji}\,x'_j$$

Here λ_{ji} is an element of the transformation matrix

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix}$$

Thus each element I_{kl} of inertia tensor $\mathbb{I}$ in a fixed coordinate system can be transformed into rotated (body) coordinates resulting in elements I_{ij} of inertia tensor $\mathbb{I}'$

The preceding result may be written as

$$I'_{ij} = \sum_{k,l} \lambda_{ik} (I_{kl} \lambda_{lj})^t$$

But $I_{kl} = I^t_{kl}$ ← symmetric matrix

$$\Rightarrow \quad I'_{ij} = \sum_{k,l} \lambda_{ik} I_{kl} \lambda^t_{lj}$$

where λ^t_{lj} are the elements of the transposed matrix $\boldsymbol{\lambda}^t$. Just as in matrix notation, we may write

$$\mathbb{I}' = \boldsymbol{\lambda} \mathbb{I} \boldsymbol{\lambda}^t$$

Since for othogonal transformations, $\boldsymbol{\lambda}^t = \boldsymbol{\lambda}^{-1}$, where $\boldsymbol{\lambda}^{-1}$ is the inverse matrix, we may write

$$\boxed{\mathbb{I}' = \boldsymbol{\lambda} \mathbb{I} \boldsymbol{\lambda}^{-1}}$$

This is the SIMILARITY TRANSFORMATION ($\mathbb{I}'$ IS SIMILAR to $\mathbb{I}$)

11-42

Let us rotate the coordinate system by an angle θ about the x_3-axis

$$\lambda = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\lambda^t = \lambda^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Observe that

$$\lambda\lambda^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Yes $\lambda\lambda^{-1} = \mathbb{1}$ as required.

Let us perform the similarity transformation on the diagonalized solid cube (pivot pt @ corner)

$$\{I\} = \begin{bmatrix} 1/6 & 0 & 0 \\ 0 & 11/12 & 0 \\ 0 & 0 & 11/12 \end{bmatrix} Mb^2$$

11-43.

$$\{I'\} = \lambda\{I\}\lambda^{-1}$$

$\alpha = 1/6\, mb^2$

$\beta = 11/12\, mb^2$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha\cos\theta & -\alpha\sin\theta & 0 \\ \beta\sin\theta & \beta\cos\theta & 0 \\ 0 & 0 & \beta \end{bmatrix}$$

$$\{I'\} = \begin{bmatrix} \alpha\cos^2\theta + \beta\sin^2\theta & \frac{1}{2}(\beta-\alpha)\sin 2\theta & 0 \\ \frac{1}{2}(\beta-\alpha)\sin 2\theta & \alpha\sin^2\theta + \beta\cos^2\theta & 0 \\ 0 & 0 & \beta \end{bmatrix}$$

We define the trace as:

$$tr\{I\} \equiv \sum_k I_{kk}$$

$$tr\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{bmatrix} = \alpha + \beta + \beta = \alpha + 2\beta \qquad (1)$$

$$tr\{I'\} = (\alpha\cos^2\theta + \beta\sin^2\theta) + (\alpha\sin^2\theta + \beta\cos^2\theta) + \beta$$

$$= \alpha(\cos^2\theta + \sin^2\theta) + \beta(\cos^2\theta + \sin^2\theta) + \beta$$

$$= \alpha + 2\beta$$

Under similarity transformations, the trace is an INVARIANT QUANTITY

$$tr\{\mathbb{I}\} = tr\{\mathbb{I}'\}$$

You can prove this in 11-22.

The determinant of the elements of a tensor is an INVARIANT QUANTITY under a similarity transformation

$$\mathbb{I}' = \lambda \mathbb{I} \lambda^{-1} =$$

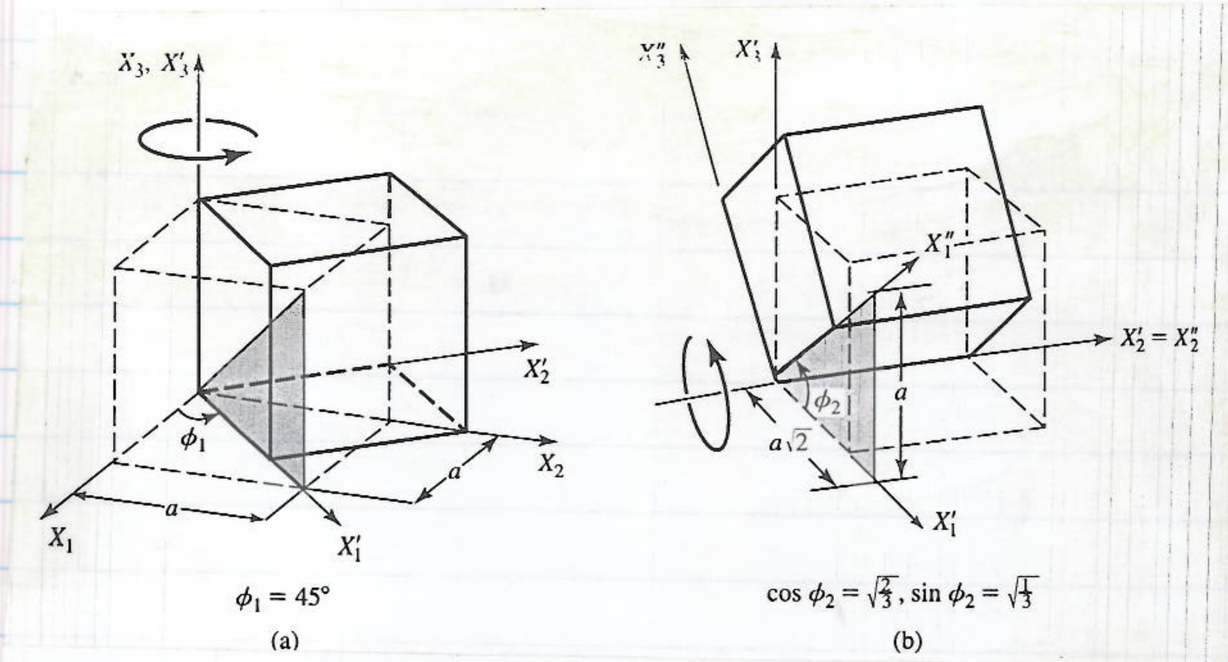
$$|\mathbb{I}'| = |\lambda \mathbb{I} \lambda^{-1}| = |\lambda| \times |\mathbb{I}| \times |\lambda^{-1}|$$
$$= |\mathbb{I}| \times |\lambda| \times |\lambda^{-1}| = |\mathbb{I}| \times |\lambda \lambda^{-1}|$$
$$= |\mathbb{I}| \times |\mathbb{1}| = |\mathbb{I}'|$$

→ For your HW I want you to verify this with the cube pivoted at the corner (DIAGONALIZED and axes along edges.) i.e.

$$\{\mathbb{I}\} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{bmatrix} mb^2 \text{ and}$$

$$\{\mathbb{I}'\} = \begin{bmatrix} 1/6 & 0 & 0 \\ 0 & 11/12 & 0 \\ 0 & 0 & 11/12 \end{bmatrix} mb^2$$

11-45



We seek to diagonalize the inertia tensor of a uniform solid cube pivoted at one corner. The initial axes are along the side of the cube. We have:

$$\{I\} = \begin{bmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{bmatrix} Ma^2$$

From our earlier analysis we found that for the diagonalized matrix, one of the principal axes is along the BODY DIAGONAL

Let us rotate the axis through an angle of $\frac{\pi}{4}$ about the x_3-axis

$$\lambda_1 = \begin{bmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} & 0 \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We now rotate the body axis through an angle of $\phi_2 = \cos^{-1}\left(\sqrt{\frac{2}{3}}\right)$ about the x_2' axis. Observe that the x_1'' is now along the body diagonal.

$$\lambda_2 = \begin{bmatrix} \cos\phi_2 & 0 & \sin\phi_2 \\ 0 & 1 & 0 \\ -\sin\phi_2 & 0 & \cos\phi_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2/3} & 0 & \sqrt{1/3} \\ 0 & 1 & 0 \\ -\sqrt{1/3} & 0 & \sqrt{2/3} \end{bmatrix}$$

The complete rotation matrix is

$$\lambda = \lambda_2 \lambda_1$$ ← Note the order!

$$\lambda = \begin{bmatrix} \sqrt{2/3} & 0 & \sqrt{1/3} \\ 0 & 1 & 0 \\ -\sqrt{1/3} & 0 & \sqrt{2/3} \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\lambda = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ -\sqrt{3}/2 & \sqrt{3}/2 & 0 \\ -\sqrt{2}/2 & -\sqrt{2}/2 & \sqrt{2} \end{bmatrix}$$

$$\lambda^{-1} = \lambda^t = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -\sqrt{3}/2 & -\sqrt{2}/2 \\ 1 & \sqrt{3}/2 & -\sqrt{2}/2 \\ 1 & 0 & \sqrt{2} \end{bmatrix}$$

← Note that for orthogonal transformations $\lambda^{-1} = \lambda^t$!!

EASY TO INVERT MATRIX!

11-47

Let $\beta = Ma^2$. Performing the similarity transformation

$$\{I'\} = \frac{\beta}{3}\begin{bmatrix} 1 & 1 & 1 \\ -\sqrt{3/2} & \sqrt{3/2} & 0 \\ -\sqrt{2}/2 & -\sqrt{2}/2 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{bmatrix}\begin{bmatrix} 1 & -\sqrt{3}/2 & -\sqrt{2}/2 \\ 1 & \sqrt{3/2} & -\sqrt{2}/2 \\ 1 & 0 & \sqrt{2} \end{bmatrix}$$

$$\{I'\} = \begin{bmatrix} \frac{1}{6}\beta & 0 & 0 \\ 0 & \frac{11}{12}\beta & 0 \\ 0 & 0 & \frac{11}{12}\beta \end{bmatrix}$$

Et voilà, the matrix is diagonalized!