Lagrangian and Hamiltonian Dynamics

By using Newton's second law and given the initial conditions we can obtain the equations of motion of the system. However Newton's laws can be used ONLY if all the forces acting on the system are known. That is to say that the DYNAMICAL CONDITIONS are known.

In most situations, even if we know the dynamical and initial conditions, it is very difficult to solve the problem. For example consider the case of a mass constrained to move on a spherical surface or a bead that slides on a wire. In these cases, NOT ONLY are the forces of constraint difficult to quantify, but making use of rectangular or other commonly used coordinate systems render the problem nearly impossible to solve.

Should we give up? Heck No!

We shall make use of HAMILTON'S PRINCIPLE:

* Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies.

In terms of the calculus of variations

$$\delta \int_{t_1}^{t_2} (T-U)\,dt = 0$$

This variational statement of the principle requires only that the integral of T-U be an <u>EXTREMUM</u>, not necessarily a <u>MINIMUM</u>

The kinetic energy of a particle expressed in fixed, rectangular coordinates is a function of $\dot{x}_i$ ONLY. And if the particle moves in a conservative force field, the potential energy is a function of position only (x_i)

$$T = T(\dot{x}_i) \quad \text{and} \quad U = U(x_i)$$

Let us define the difference between these quantities to be:

$$L \equiv T - U = L(x_i, \dot{x})$$

Then Hamilton's Principle is

$$\boxed{\delta \int_{t_1}^{t_2} L(x_i, \dot{x}_i)\,dt = 0}$$

This function L may be identified with the function f in the variational integral (i.e. calculus of variations)

Let us make the transformations: [must be a conservative field, i.e. $F = -\nabla U$]

$$x \to t$$
$$y_i(x) \to x_i(t)$$
$$y_i'(x) \to \dot{x}_i(t)$$

[N.B. $\frac{dx}{dt} = \dot{x}$]

$$f\{y_i(x), y_i'(x); x\} \longrightarrow L(x_i, \dot{x}_i)$$

The Euler-Lagrange equations become

$$\boxed{\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0 \qquad i = 1, 2, 3}$$

These are the LAGRANGE EQUATIONS of MOTION for the particle. The quantity L is called the LAGRANGE FUNCTION or the LAGRAGIAN for the particle.

Again:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

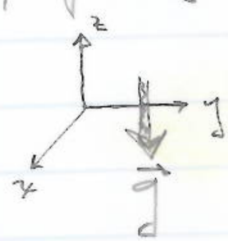
$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0$$

$$\frac{\partial L}{\partial z} - \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = 0$$

Examples

Let us calculate the equations of motion using Hamilton's Principle for a single particle of mass m moving under the influence of gravity

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$



$$U = mgz$$

$$L = T - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

$$\begin{cases} \frac{d}{dt}(m\dot{x}) = 0 \\ \frac{d}{dt}(m\dot{y}) = 0 \\ \frac{d}{dt}(m\dot{z}) + mg = 0 \end{cases} \Rightarrow \begin{cases} \dot{x} = \text{const} \\ \dot{y} = \text{const} \\ \ddot{z} = -g \end{cases}$$

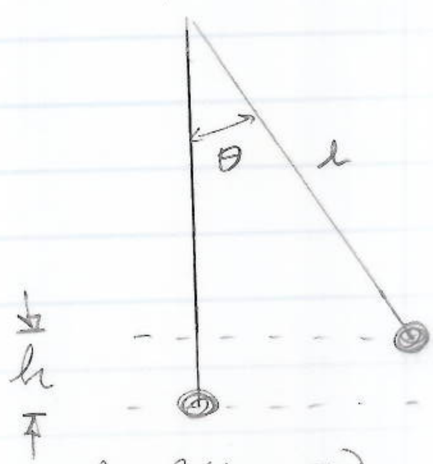
Now let's try something a bit more complicated

$$T = \frac{1}{2}m(l\dot{\theta})^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

$$U = mgl(1 - \cos\theta)$$

Here we have let $\dot{x} \rightarrow \dot{\theta}$

$x \rightarrow \theta$



$h = l(1 - \cos\theta)$

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1-\cos\theta)$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) \Rightarrow \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

For θ small, $\ddot{\theta} + \frac{g}{l}\theta = 0$
and the angular frequency is $\omega^2 = \frac{g}{l}$.

Another Example

Let us find the equations of motion of a particle in terms of the polar coordinate variables r and θ

The element of arc is $ds^2 = dr^2 + r^2 d\theta^2$
and the velocity is $v^2 = \left(\frac{ds}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2$

$$= \dot{r}^2 + r^2\dot{\theta}^2$$

We can also derive this by letting

$x = r\cos\theta$ and $y = r\sin\theta$ ← relationship between rectangular and polar coordinates.

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$
$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2\cos^2\theta - 2r\dot{r}\dot{\theta}\cos\theta\sin\theta + r^2\dot{\theta}^2\sin^2\theta$$
$$+ \dot{r}^2\sin^2\theta + 2r\dot{r}\dot{\theta}\cos\theta\sin\theta + r^2\dot{\theta}^2\cos^2\theta$$

$$= \dot{r}^2 + r^2\dot{\theta}^2$$

Now let us require that the particle is subject to an inverse-square attractive force (and we note that the force is conservative - which means that $\vec{F} = -\nabla U$)

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$U = -\frac{k}{r}$$

The Lagragian in polar coordinates is:

$$L = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}$$

in Lagrange's Equations:

(1) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

(2) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

Let us examine equation (1)

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}, \text{ and } \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{k}{r^2}$$

This means.

$$m\ddot{r} - mr\dot{\theta}^2 = -\frac{k}{r^2}$$

$$F_r = -\frac{\partial U}{\partial r}$$

Hence

$$m\ddot{r} = mr\dot{\theta}^2 + F_r \qquad (3)$$

Now from equation (2)

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = 0$$

Hence Lagrange's Equation for $\theta, \dot{\theta}$ takes the form

$$\frac{d}{dt}\underbrace{(mr(r\dot{\theta}))}_{} = 0$$

$J = mr^2\dot{\theta}$ = angular momentum.

$\Rightarrow$ $J = $ const [Is $L = L(\theta)$? Then it is <u>cyclic</u> in θ]

We may therefore conclude that in a conservative field, the <u>angular momentum</u> is a <u>constant</u> of the motion and the <u>torque</u> is <u>ZERO</u>.

GENERALIZED COORDINATES

In the preceding we have made use of rectangular and polar coordinates, i.e.

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

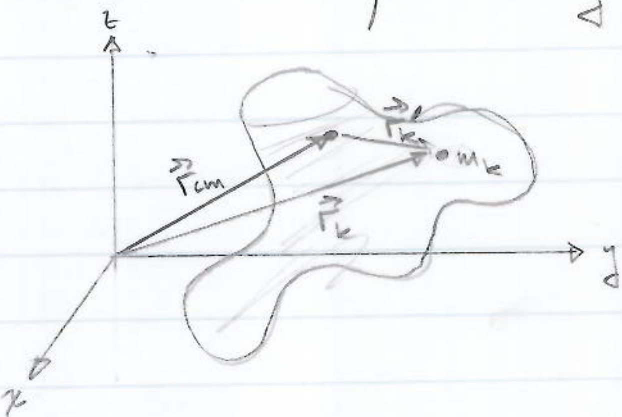
$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$

This suggest a certain flexibility in specifying coordinates that is internal to the formalism of the Lagrange Equations

Let us consider a mechanical system consisting of N particles. To specify the position of such a system at any given time we need N vectors and each vector is described by three coordinates. In general, we NEED 3N coordinates to describe a given mechanical system. If there constraints, the <u>total number</u> of coordinates needed to specify the system will be <u>reduced</u>. As was explained last semester, a rigid body can be completely described by only SIX COORDINATES; that is, only six coordinates are needed to specify the configuration of a rigid body.

Of these six, three coordinates give the position of some convenience reference point in the body – usually the center of mass wrt the origin of some chosen coordinate system – and the remaining three coordinates describe the orientation of the body in space



We are interested in finding the minimum number of coordinates needed to describe a system of N particles. Usually the constraints on any given system are described by means of equations. Suppose there are m such equations that describe the constraints. The minimum number of coordinates, n, needed to completely describe the motion or the configuration of such a system at any given time is

$$n = 3N - m$$

where n is the <u>number of degrees</u> of the system. It is not necessary that these n coordinates be rectangular, cylindrical, or any other curvilinear coordinate system.

Indeed, n CAN BE ANY PARAMETER, such as length, $(\text{length})^2$, angle, energy, even a dimensionless quantity – as long as it completely describes the configuration of the system.

We give the name GENALIZED COORDINATES to any set of quantities that completely specifies the state of a system.

The n generalized coordinates are customarily written as

$$q_1, q_2, q_3, \ldots, q_n$$

or

$$q_k, \text{ where } k = 1, 2, 3, \ldots, n$$

The n generalized coordinates are not restricted by any constraints. If each coordinate can vary independently of the other, the system is said to be HOLONOMIC. In a NONHOLONOMIC system, the coordinates CANNOT vary independently. Hence in such a system, the number of degrees of freedom is less than the minimum number of coordinates needed to specify the configuration of the system. For example, let us consider a sphere constrained to roll on a plane. We need only five coordinates to specify its configuration;

two for the position of its center of mass and three for its orientation. These five coordinates, however, cannot all vary independently. - When the sphere rolls at least two coordinates must change - Hence this a nonholonomic system.

A suitable set of generalized coordinates of a system is one in which results in an easy interpretation of the motion. The q_n generalized coordinates form a CONFIGURATION SPACE, with each dimension represented by a coordinate q_k. The path in configuration space does not lend itself to the same interpretation as a path in ordinary three-dimensional space.

In analogy with Cartesian coordinates, we may define the derivatives of q_k, i.e. $\dot{q}_1, \dot{q}_2, \dot{q}_3, \ldots, \dot{q}_n$ as GENERALIZED VELOCITIES.

Generalized Forces

Single Particle:

Consider a force $\vec{F}$ that is acting on a single particle of mass m and produces a virtual displacement $\delta\vec{r}$ of the particle

Nonholonomic system

From Wikipedia, the free encyclopedia.

In physics and mathematics, a **nonholonomic system** is a system in which a return to the original internal configuration does not guarantee return to the original system position. In other words, unlike with a holonomic system, the outcome of a nonholonomic system is path-dependent.

For example, when riding a two-wheeled cart, a return to the original internal (wheel) configuration does not guarantee return to the original system (cart) position.

Cars, bicycles and unicycles are all examples of nonholonomic systems.

The branch of mathematics dealing with nonholonomic systems is known as sub-Riemannian geometry.

External links

- A comparison of two systems (http://www.nd.edu/NDInfo/Research/sskaar/Comparison.html)

Retrieved from "http://en.wikipedia.org/wiki/Nonholonomic_system"

Categories: Physics stubs | Algebraic topology | Differential topology | Classical mechanics

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The work done δW by this force is

$$\delta W = \vec{F} \cdot \delta \vec{r} = F_x \delta x + F_y \delta y + F_z \delta z$$

We can express the displacements δx, δy, and δz in terms of the generalized coordinates:

$$\delta x = \frac{\partial x}{\partial q_1} \delta q_1 + \frac{\partial x}{\partial q_2} \delta q_2 + \cdots + \frac{\partial x}{\partial q_k} \delta q_k$$

$$= \sum_{k=1}^{n} \frac{\partial x}{\partial q_k} \delta q_k$$

with similar expressions for δy and δz

δW may now be written

$$\delta W = \sum_{k=1}^{n} \left(F_x \frac{\partial x}{\partial q_k} + F_y \frac{\partial y}{\partial q_k} + F_z \frac{\partial z}{\partial q_k} \right) \delta q_k$$

$$= \sum_{k=1}^{n} Q_k \delta q_k$$

where Q_k is called the generalized force

$$Q_k = F_x \frac{\partial x}{\partial q_k} + F_y \frac{\partial y}{\partial q_k} + F_z \frac{\partial z}{\partial q_k}$$

The dimensions of Q_k depend on the dimensions of q_k. The dimensions of $Q_k \delta q_k$ is that of work. If the increment δq_k has the dimensions of distance, Q_k will have the dimension of force.

If the increment δq_k has the dimensions of angle θ, Q_k will have dimensions of torque T_θ. It must be pointed out that the quantity δq_k and the quantities δx, δy, δz are called virtual displacements of the system because it is not necessary that such displacements represent any actual displacements.

LAGRANGE'S EQUATIONS of MOTION in GENERALIZED COORDINATES

We are interested in describing the motion of a single particle by means of equations written in terms of generalized coordinates. This leads to Lagrange's Equations.

Let us start with an expression for the kinetic energy T written in terms of cartesian coordinates

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

since $x = x(q_1, q_2, \ldots, q_n) = x(q)$
similarly $y = y(q)$ and $z = z(q)$

We can evaluate $\dot{x}$ in terms of q_k by the following procedure

$$\dot{x} = \frac{\partial x}{\partial q_1}\frac{\partial q_1}{\partial t} + \frac{\partial x}{\partial q_2}\frac{\partial q_2}{\partial t} + \cdots + \frac{\partial x}{\partial q_n}\frac{\partial q_n}{\partial t}$$

(1) $$= \sum_{k=1}^{n} \frac{\partial x}{\partial q_k}\frac{\partial q_k}{\partial t} = \sum_{k=1}^{n} \frac{\partial x}{\partial q_k}\dot{q}_k = \dot{x}(q,\dot{q})$$

Thus we can describe the different components of velocity in terms of the generalized coordinates q_k and the generalized velocities $\dot{q}_k$

$$\dot{x} = \dot{x}(q,\dot{q}), \quad \dot{y} = \dot{y}(q,\dot{q}), \quad \dot{z} = \dot{z}(q,\dot{q})$$

We may now write the kinetic energy as

$$T = \frac{1}{2}m\left[\dot{x}^2(q,\dot{q}) + \dot{y}^2(q,\dot{q}) + \dot{z}^2(q,\dot{q})\right]$$

Taking the derivative wrt the generalized velocity $\dot{q}_k$

$$\frac{\partial T}{\partial \dot{q}_k} = m\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_k} + \dot{y}\frac{\partial \dot{y}}{\partial \dot{q}_k} + \dot{z}\frac{\partial \dot{z}}{\partial \dot{q}_k}\right) \qquad (2)$$

Using equation (1) i.e. differentiating (1) wrt $\dot{q}$ $\Rightarrow$

$$\frac{\partial \dot{x}}{\partial \dot{q}_k} = \frac{\partial x}{\partial q_k} \qquad (3)$$

Note that $\partial x/\partial q_k$ is the coefficient of $\dot{q}_k$ in the expression for $\dot{x}$ in equation (1).

Substituting (3) into (2) gives

$$\frac{\partial T}{\partial \dot{q}_k} = m\left(\dot{x}\frac{\partial x}{\partial q_k} + \dot{y}\frac{\partial y}{\partial q_k} + \dot{z}\frac{\partial z}{\partial q_k}\right) \qquad (4)$$

Now differentiating both sides of equ. (4) wrt t:

(5) $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = m\ddot{x}\frac{\partial x}{\partial q_k} + m\ddot{y}\frac{\partial y}{\partial q_k} + m\ddot{z}\frac{\partial z}{\partial q_k} + m\dot{x}\frac{d}{dt}\left(\frac{\partial x}{\partial q_k}\right) + m\dot{y}\frac{d}{dt}\left(\frac{\partial y}{\partial q_k}\right) + m\dot{z}\frac{d}{dt}\left(\frac{\partial z}{\partial q_k}\right)$$

To simplify the last three terms on the RHS, we make use of the fact that d/dt and $\partial/\partial q_k$ are interchangeable

$$\frac{d}{dt}\left(\frac{\partial x}{\partial q_k}\right) = \frac{\partial}{\partial q_k}\left(\frac{dx}{dt}\right) = \frac{\partial \dot{x}}{\partial q_k} \qquad (6)$$

We see that from (6), the fourth term in (5) can be written.

$$m\dot{x}\frac{d}{dt}\left(\frac{\partial x}{\partial q_k}\right) = m\dot{x}\frac{\partial \dot{x}}{\partial q_k} = \frac{\partial}{\partial q_k}\left(\frac{1}{2}m\dot{x}^2\right) \qquad (7)$$

with similar expressions for other terms. Also note that

$$F_x = m\ddot{x} \quad F_y = m\ddot{y} \text{ and } F_z = m\ddot{z}$$

Equation (5) becomes

$$(8) \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = F_x \frac{\partial x}{\partial q_k} + F_y \frac{\partial y}{\partial q_k} + F_z \frac{\partial z}{\partial q_k} + \frac{\partial}{\partial q_k}\left[\frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\right]$$

Now using the definition of generalized force and kinetic energy

$$Q_k = F_x \frac{\partial x}{\partial q_k} + F_y \frac{\partial y}{\partial q_k} + F_z \frac{\partial z}{\partial q_k}$$

and

$$T = \frac{1}{2} m \left[\dot{x}^2(q,\dot{q}) + \dot{y}^2(q,\dot{q}) + \dot{z}^2(q,\dot{q})\right]$$

and inserting into eqv. (8) gives

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = Q_k + \frac{\partial T}{\partial q_k}}$$

These differential equations in generalized coordinates describe the motion of a particle and are known as

LAGRANGE'S EQUATIONS
of MOTION

Now from our example on a mass m moving in a plane and subject to an inverse-square law ($\vec{F} = -\nabla U$) $\Rightarrow U = -\frac{k}{r}$

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

Now $$Q_k = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}$$

$$\frac{\partial T}{\partial r} = mr\dot{\theta}^2 \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) = \frac{d}{dt}(m\dot{r}) = m\ddot{r}$$

$$\boxed{F_r = m\ddot{r} - mr\dot{\theta}^2}$$

$$\frac{\partial T}{\partial \theta} = 0 \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{d}{dt}(mr^2\dot{\theta})$$

$$\frac{d}{dt}(mr^2\dot{\theta}) = Q_\theta = \tau$$

But we found from the Lagrangian $L = \frac{1}{2}m(\dot{r}^2 - r^2\dot{\theta}^2) + \frac{k}{r}$ That

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0$$

$$mr^2\dot{\theta} = mr^2\omega = r(mv) = |\vec{r} \times \vec{p}|$$

$\vec{r} \perp \vec{p}$

The angular momentum is conserved

$$\Rightarrow \underline{Q_\theta = \tau = 0}$$

Lagrange's equations assume a much simpler form when the motion is in a CONSERVATIVE FORCE field so that

$$Q_k = -\frac{\partial U}{\partial q_k}$$

Substituting this into the above equation yields

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \frac{\partial T}{\partial q_k} - \frac{\partial U}{\partial q_k}$$

The Lagrange Function is defined $L = T - U$

$$L = T - U \quad \text{or} \quad L(q, \dot{q}) = T(q, \dot{q}) - U(q)$$

since $U = U(q)$ $\quad \frac{\partial U}{\partial \dot{q}_k} = 0$

[if U is not independent of velocity $\dot{q}$, then $U = U(q, \dot{q})$ will lead to a tensor force]

$$\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k}(T - U) = \frac{\partial T}{\partial \dot{q}_k}$$

$$\frac{\partial L}{\partial q_k} = \frac{\partial}{\partial q_k}(T - U) = \frac{\partial T}{\partial q_k} - \frac{\partial U}{\partial q_k}$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0}$$

(For a conservative force field)