Newton's Laws of Motion

Sir Isaac Newton was the first to give a complete formulation of the laws of mechanics.

1. Every body continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it.

2. Rate of change of momentum is proportional the impressed force, and is in the direction in which the force acts. $\vec{F} = \frac{d\vec{p}}{dt}$.

3. To every action there is always an equal and opposite reaction.

The second law is a differential equation that relates the time rate of change of the linear momentum to the force. It is really a definition.

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v})$$

if the mass is not changing then (and only then)

$$\vec{F} = m\frac{d^2\vec{x}}{dt^2} = m[\ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}]$$

Newton's Third Law states that

$$\vec{F}_1 = -\vec{F}_2$$

$$\frac{d\vec{p}_1}{dt} = -\frac{d\vec{p}_2}{dt}$$

with constant masses

$$m_1 \frac{d\vec{v}_1}{dt} = -m_2 \frac{d\vec{v}_2}{dt} \Rightarrow m_1(\vec{a}_1) = m_2(-\vec{a}_2)$$

Hence

$$\frac{m_2}{m_1} = -\frac{\vec{a}_1}{\vec{a}_2}$$

; the negative sign indicates only that the acceleration vectors are oppositely directed

Restatement of NIII

Another interpretation of Newton's Third Law is:

$$\frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = 0$$

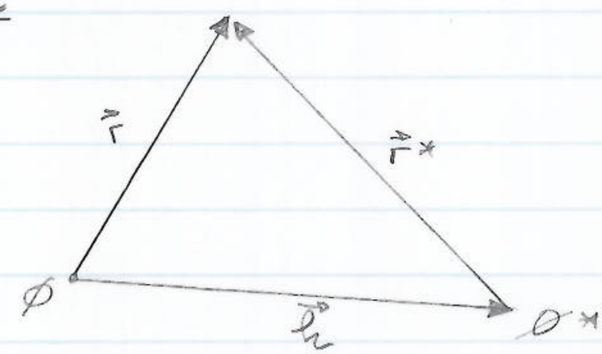
or

$$\vec{p}_1 + \vec{p}_2 = \text{constant}$$

Linear momentum is conserved in the isolated interaction of two particles.

$\boxed{\text{CONSERVATION OF LINEAR MOMENTUM}}$ ← In General.

Frames of Reference



Let a point in space be located by vectors $\vec{r}$, $\vec{r}^*$ wrt two origins of coordinates $\emptyset$, $\emptyset^*$, and let $\emptyset^*$ be located by a vector $\vec{h}$ wrt $\emptyset$. $\Rightarrow$ $\vec{r} = \vec{r}^* + \vec{h}$

$$x = x^* + h_x \qquad y = y^* + h_y \qquad z = z^* + h_z$$

Now if the origin $\emptyset^*$ is moving wrt the origin $\emptyset$, which we regard as fixed, then

$$\vec{V} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}^*}{dt} + \frac{d\vec{h}}{dt} = \vec{V}^* + \vec{V}_h$$

This is an example of a <u>translation</u> of the starred coordinate system wrt the unstarred one.

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2\vec{r}^*}{dt^2} + \frac{d^2\vec{h}}{dt^2} = \vec{a}^* + \vec{a}_h$$

Newton's Laws of motion hold in a fixed coordinate system, so that we have, for a particle of mass m subject to a force $\vec{F}$

$$\Longrightarrow \quad \vec{F} = m\frac{d^2\vec{r}}{dt^2} = m\frac{d^2\vec{r}^*}{dt^2} + m\frac{d^2\vec{h}}{dt^2}$$

If ϕ^* is moving at <u>constant velocity</u> relative to ϕ, then

$$\frac{d^2\vec{h}}{dt^2} = 0$$

and we have

$$\vec{F} = m\frac{d^2\vec{r}^*}{dt^2}$$

Thus, if Newton's Laws of motion hold in one coordinate system, these Laws will hold in another coordinate system moving at uniform velocity relative to the first. This is called the PRINCIPLE OF NEWTONIAN RELATIVITY.

Now let's do some elementary problems in mechanics....

Let $m \neq m(t)$ then

$$\vec{F} = m\vec{a}$$

$$a = \frac{dv}{dt} = \frac{F}{m}$$

$$dv = \frac{F}{m}dt \qquad \begin{matrix} F \neq F(t) \\ a \neq a(t) \end{matrix}$$

$$\int_{v_0}^{v} dv = \int_0^t \frac{F}{m}dt \Rightarrow v - v_0 = \frac{F}{m}t$$

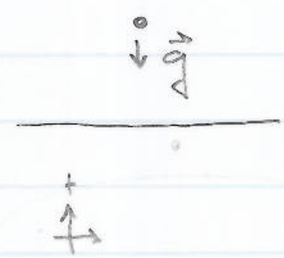
$$v = \frac{dx}{dt} = v_0 + \frac{F}{m}t \quad \Leftarrow$$

$$\int_{x_0}^{x} dx = \int_0^t \left(v_0 + \frac{F}{m}t\right) dt$$

$$\Rightarrow x = x_0 + v_0 t + \frac{1}{2}\frac{F}{m}t^2$$

A body falling freely near the surface of the earth is acted upon by a constant force

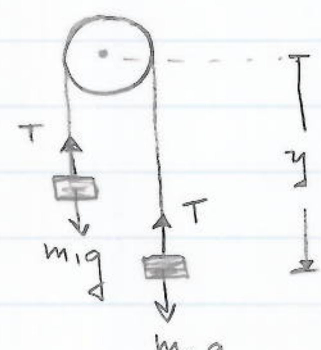
$$\vec{F} = m_0 \underbrace{\left(G\frac{M}{R^2}\right)}_{\vec{g}} \hat{r}$$



The earth appears flat near the surface

$$a_y = -g$$
$$v_y = v_{y_0} - gt$$
$$y = y_0 + v_{y_0}t - \frac{1}{2}gt^2$$

Atwood's Machine



Since the length of the rope is constant, the coordinate y fixes the positions of both m_1 and m_2. Both move with the same speed:

$$v = \frac{dy}{dt}$$

$$\sum F_1 = -m_1 g + T = m_1 a$$

$$\sum F_2 = -m_2 g + T = -m_2 a$$

where $a = \frac{dv}{dt}$ and the magnitude of the acceleration is the same for both masses.

$$T = m_1(a+g) = m_2(g-a)$$

$$\Rightarrow a = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) g$$

HW

$$\Rightarrow T = \frac{2m_1 m_2}{m_1 + m_2} g$$

As a check: if $m_1 = m_2$, $a = 0$ & $T = m_1 g = m_2 g$
(The masses are in static equilibrium)

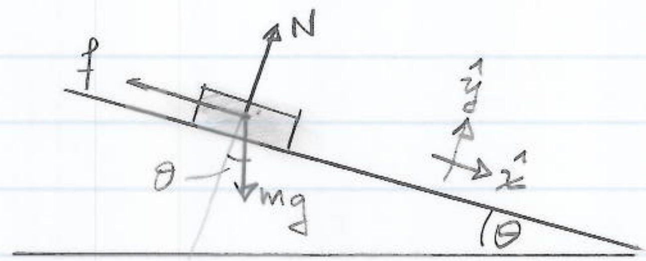
If $m_2 \gg m_1$

$$a \simeq g \quad \& \quad T \simeq 2m_1 g$$

HW explain why?

2-7

Brick sliding down an inclined plane having a rough surface



(1) $\sum F_x = mg\sin\theta - f = ma$

(2) $\sum F_y = N - mg\cos\theta = 0 \Rightarrow N = mg\cos\theta$

The frictional force is proportional to the normal force N

(3) $f = \mu_k N = \mu_k (mg\cos\theta)$

$\uparrow$ coefficient of kinetic friction

(3) → (1)

(4) $\ddot{x} = g(\sin\theta - \mu_k\cos\theta)$; acceleration is cnst.

we can now find the velocity of the block from rest a distance x_0 down the plane by multiplying equ. (4) by $2\dot{x}$ and integrating

$$2\dot{x}\ddot{x} = 2\dot{x}g(\sin\theta - \mu_k\cos\theta)$$

$$\frac{d}{dt}(\dot{x}^2) = 2g(\sin\theta - \mu_k\cos\theta)\frac{dx}{dt}$$

$$\int_0^{V_0^2} d(\dot{x}^2) = 2g(\sin\theta - \mu_k\cos\theta)\int_0^{x_0} dx$$

$$V_0^2 = 2g(\sin\theta - \mu_k\cos\theta)\,x_0$$

$$V_0 = \left[2g(\sin\theta - \mu_k\cos\theta)\,x_0\right]^{1/2}$$

If the brick is at rest, the frictional force f may have any value up to a maximum of $\mu_s N$

$$f \leq \mu_s N$$

where μ_s, the coefficient of static friction, is usually greater than μ_k. In the case of $a = 0$ in equ. (1) we obtain

$$f = mg\sin\theta \leq \mu_s mg\cos\theta$$

According to the above relation, the angle θ of the incline must not be greater than a limiting value θ_r, the angle of repose

$$\tan\theta \leq \tan\theta_r = \mu_s.$$

If $\theta > \theta_r$, the brick cannot remain at rest.

We found earlier in polar coordinates

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\theta}$$

If a body moves with constant v_θ around a circle of radius r, its acceleration is towards the center of the circle $(\dot{r} = \ddot{r} = \ddot{\theta} = 0)$

$$\vec{a} = -r\dot{\theta}^2\,\hat{r}$$

$$\dot{\theta} = \frac{v_\theta}{r} \Rightarrow |\vec{a}| = \frac{v_\theta^2}{r}$$

(1) $F = ma = \dfrac{mv^2}{r}$

(2) $F = G\dfrac{Mm}{r^2}$; $g = \dfrac{GM}{R^2} \Rightarrow GM = gR^2$

(3) $F = mg\dfrac{R^2}{r^2}$

Now $v_\theta = \dfrac{2\pi r}{T}$ (4) $T \leftarrow$ period of revolution

Equating (1) and (3) and substituting equ. (4) :

$\rightarrow$ $r^3 = \dfrac{gR^2}{4\pi^2}T^2$ $\leftarrow$ Kepler's Third Law.

Let $g = 9.8\,\mathrm{m/s^2}$ $R = 6{,}368$ km $T = 27\tfrac{1}{3}$ days.
We obtain the earth-moon distance: $r = 383{,}000$ km
According to modern measurements: $r = 385{,}000$ km

2-9'

Note that $\left| \frac{383k - 385k}{385k} \right| \leq \underline{1\%}$!

Also note

$$\underline{r^3 = K T^2}$$

The Earth is $\underline{1 \text{ A.U.}}$ distant from the sun and its period is 1 yr.

$$\Rightarrow K = \frac{(1 \text{ AU})^3}{(1 \text{ yr})^2}$$

Now Jupiter is: 5.187 AU.

$$T = \left[\frac{(5.187 \text{ AU})^3}{K} \right]^{1/2} \Rightarrow 11.81 \text{ yr.}$$

The actual value is 11.86 yr. $\Rightarrow \left| \frac{11.81 - 11.86}{11.86} \right| < 0.5\%$!

Q: If $r = 100$ AU what is T?

A. $\underline{T = 1000 \text{ yr}}$

Damping Force depending on the Velocity

When F is a function of the velocity alone:

$$m\frac{dv}{dt} = F(v)$$

(1) $$\int_{v_0}^{v} \frac{dv}{F(v)} = \frac{t-t_0}{m}$$

In certain cases and over certain ranges of velocity, the frictional force is proportional to some fixed power of the velocity

$$F = (\mp) b v^n$$ ← If n is an odd integer then the negative sign should be chosen

The sign must be chosen so that the force has the opposite sign to the velocity v. The frictional force is always opposed to the velocity, and therefore does negative work, i.e., absorbs energy from the moving body

Let $n=1 \Rightarrow m\frac{dv}{dt} = -bv = F(v)$

inserting this into equ (1)

$$\int_{v_0}^{v} \frac{dv}{v} = -\frac{b}{m}t$$

$$\ln\frac{v}{v_0} = -\frac{b}{m}t \Rightarrow v = v_0 e^{-\frac{b}{m}t}$$

2-11

We see that as $t \to \infty$, $v \to 0$. To find the displacement

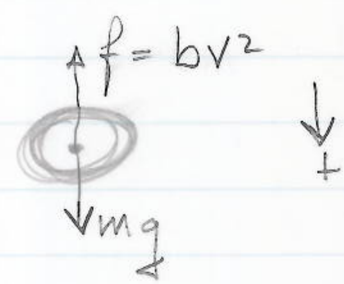
$$x = \int_0^t v_0 e^{-\frac{b}{m}t}\,dt = \frac{mv_0}{b}\left(1 - e^{-\frac{b}{m}t}\right)$$

As $t \to \infty$, $x \to \frac{mv_0}{b}$

Ex

Find the displacement of a particle undergoing vertical motion in a medium having a retarding force proportional to the square of its velocity
[particle is projected downward]

$$\Sigma F_y = mg - bv^2 = m\frac{dv}{dt}$$



$$\frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy} = g - \frac{b}{m}v^2$$

$$\int_{v_0}^{v_1} \frac{v\,dv}{g - \frac{b}{m}v^2} = \Delta y$$

Let $u = g - \frac{b}{m}v^2$ $\quad du = -2\frac{b}{m}v\,dv$

$$\Rightarrow v\,dv = -\frac{1}{2}\frac{m}{b}du$$

$$-\frac{1}{2}\frac{m}{b}\int \frac{du}{u} = \Delta y$$

$$\Rightarrow \Delta y = \frac{1}{2}\frac{m}{b}\ln\left[\frac{g - \frac{b}{m}v_0^2}{g - \frac{b}{m}v_1^2}\right]$$

Let us project the particle downward with an initial speed of 0. (i.e. drop it) What is its terminal velocity V_t?

$$y = \frac{1}{2}\frac{m}{b}\ln\left[\frac{g}{g - \frac{b}{m}v^2}\right]$$

Rewriting

$$-y = \frac{1}{2}\frac{m}{b}\ln\left[\frac{g - \frac{b}{m}v^2}{g}\right]$$

$$\frac{g - \frac{b}{m}v^2}{g} = e^{-2\frac{b}{m}y}$$

$$\Rightarrow V^2 = \frac{m}{b}g\left(1 - e^{-2\frac{b}{m}y}\right)$$

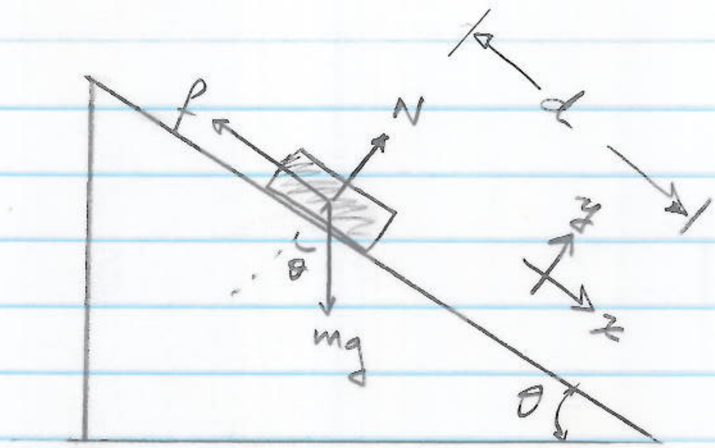
As $y \to \infty$ $V \to V_t$ $\Rightarrow$

$$V_t^2 = \frac{m}{b}g$$

$$V_t = \left[\frac{m}{b}g\right]^{1/2}.$$

2-13

Brick slides down an inclined plane under the influence of gravity. The force opposing its motion is $f = kmv^2$



Find its speed as a function of t and then determine the time that is required to travel a distance d.

We shall assume that it starts from rest

$$\Sigma F_x = mg\sin\theta - f = m\frac{dv}{dt}$$

$$f = mkv^2$$

$$mg\sin\theta - mkv^2 = m\frac{dv}{dt}$$

$$\int_0^{v(t)} \frac{dv'}{g\sin\theta - kv'^2} = \int dt$$

$$t = \frac{1}{k}\int_0^v \frac{dv'}{\frac{g}{k}\sin\theta - v'^2} = \frac{1}{k}\int_0^v \frac{dv'}{V_t^2\sin\theta - V'^2}$$

$$V_t^2 = g/k$$

$$\boxed{\begin{aligned} a^2 &= V_t^2 \sin^2\theta \\ b^2 &= -1 \Rightarrow b = i \end{aligned}}$$

We now have an integral of the form

$$\int_0^v \frac{dv'}{(b^2v^2+a^2)} = -\int_v^0 \frac{dv'}{a^2+b^2v^2} \qquad \begin{aligned} a^2 &= V_t^2 \sin\theta \\ b^2 &= -1 \leftarrow b = i \end{aligned}$$

We observe that $\displaystyle\int \frac{dx}{a^2+b^2x^2} = \frac{1}{ab}\tan^{-1}\frac{bx}{a}$

$\tan^{-1} icx = \tanh^{-1} cx \;,\; c = \frac{|b|}{a}$

[Note that $\cos(iz) = \cosh(z)$; $i\sin(iz) = \sinh(z)$
(see problem 2:33)

$\tan^{-1}(iz) \Rightarrow \tanh^{-1}(z) \Leftarrow$ Prove this.]

From the integral table, we have $t = \frac{1}{k}\frac{1}{\sqrt{V_t^2 \sin\theta}}\tanh^{-1}\left(\frac{v}{V_t}\frac{1}{\sqrt{\sin\theta}}\right)$

Solving for V we obtain

$$V = V_t \sin^{1/2}\theta \tanh\left[(kV_t \sin^{1/2}\theta)t\right] = \frac{dx}{dt}$$

And $\displaystyle\int_0^d dx = V_t \sin^{1/2}\theta \int_0^t \tanh\left[(kV_t \sin^{1/2}\theta)t'\right]dt'$

(We observe that $\int \tanh x\,dx = \ln(\cosh x)$)

Making the proper substitutions and solving for t gives

$$\boxed{t = \frac{\cosh^{-1}(e^{kd})}{\sqrt{kg\sin\theta}}}$$

This will be a HW problem.

We now have an integral in the form of

$$\int \frac{dv'}{a^2 v^2 - b^2} = -\int \frac{dv'}{b^2 + a^2 v^2} \qquad -\int_v^0 \frac{dx}{b^2 + a^2 x^2} \Rightarrow \frac{1}{ab}\tan^{-1}\frac{bx}{a}$$

$b^2 = \sqrt{V_t^2 \sin\theta}$ $\quad a^2 = -1$

$V_t^2 \sin\theta > V$ $\quad a = i$

use an integral table to show

$$t = \frac{1}{k}\frac{1}{\sqrt{V_t^2 \sin\theta}} \tanh^{-1}\left(\frac{V}{V_t}\frac{1}{\sqrt{\sin\theta}}\right)$$

Solve for v to show

$$\Rightarrow \quad V = V_t \sin^{1/2}\theta \tanh\left[(kV_t \sin^{1/2}\theta)\, t\right]$$

$$\int_0^d dx = V_t \sin^{1/2}\theta \int_0^t \tanh\left[(kV_t \sin^{1/2}\theta)\, t'\right] dt'$$

making the proper substitutions, using E.17b in Appendix E, and solving for t yields

$$\boxed{t = \frac{\cosh^{-1}(e^{kd})}{\sqrt{kg \sin\theta}}}$$

2.29 $V = V_{term} \sin^{1/2}\theta \tanh\left[(k v_t \sin^{\frac{1}{2}}\theta)\, t\right]$.

$\theta = \frac{\pi}{2} \quad k = \frac{g}{V_{term}}$ (see eqv. 2.57)

$V = V_{term} \tanh\left(\frac{gt}{V_{term}}\right)$ $\quad V_t = 50\,m/s$ $(115\,m/s)$

time (s)	0	1	5	10	20	30
actual speed (m/s)	0	9.7	38	48	50	50
speed in vacuum (m/s)	0	9.8	49	98	196	294

2.30 We have from Eq. (2.59) $V_{term} = \left[\frac{mg}{\gamma D^2}\right]^{1/2}$

$$D = \frac{1}{V_{term}}\left[\frac{mg}{\gamma}\right]^{1/2}$$

$$= \frac{1}{50\,m/s}\left[\frac{(70\,kg)(9.8\,m/s^2)}{0.25\,kg/m^3}\right]^{1/2}$$

$D \sim 1\,m.$ $\longleftarrow$ Not so crazy.

Motion of a charged particle in a uniform $\vec{B}$ field

$$\vec{F} = q\vec{E} + q\underbrace{\vec{v}\times\vec{B}}_{\text{depends on } \vec{v}\,!} \quad ; \quad \vec{E} = 0$$

Ask. What is the work done by the B-field?
$W = \Delta K$; $\Delta K = 0 \Rightarrow W = 0$

$$m\dot{\vec{v}} = q\vec{v}\times\vec{B}$$

Let $\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$ and $\vec{B} = B\hat{z}$

$$\vec{v}\times\vec{B} = v_yB\hat{x} - v_xB\hat{y}$$

$$\begin{cases} m\dot{v}_x = qv_yB \\ m\dot{v}_y = -qv_xB \end{cases} \leftarrow \text{Transverse velocity components}$$

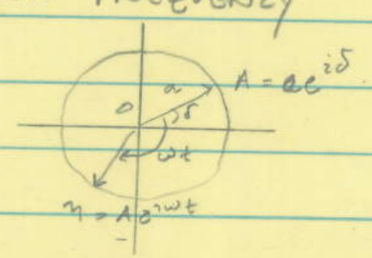
$m\dot{v}_z = 0$ ← This says the component of the particle's velocity in the direction of $\vec{B}$ is constant.

We shall focus on the transverse velocity $v_\perp$ which is $\perp$ to the direction of $\vec{B}$

Letting $\omega = \dfrac{qB}{m}$ ← CYCLOTRON FREQUENCY

The transverse components become

$$\begin{cases} \dot{v}_x = \omega v_y \\ \dot{v}_y = -\omega v_x \end{cases} \quad \text{Two coupled diff. eqns.}$$



$$v_x + iv_y = Ae^{-i\omega t} = ae^{i\delta}e^{-i\omega t} = ae^{i(\delta-\omega t)} = ae^{-i(\omega t-\delta)}$$

$$= a\cos(\omega t-\delta) - i\sin(\omega t-\delta)$$

$v_x = a\cos(\omega t-\delta)$
$v_y = -a\sin(\omega t-\delta)$

The two components of $\vec{v}_\perp$ oscillate at the same freq. but 90° out of phase.
$\vec{v}_\perp$ rotates steadily clockwise w/ const magnitude.

(2.53) A charged particle of mass m and positive charge q moves in uniform electric fields and magnetic fields

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \quad \text{LORENTZ FORCE} \quad \vec{E} \parallel \vec{B} \parallel \hat{z}$$

Eqns of motion

$$\left.\begin{aligned} m\dot{v}_x &= qBv_y \\ m\dot{v}_y &= -qBv_x \end{aligned}\right\}$$ The transverse position moves clockwise around a circle at constant angular velocity $\omega = qB/m$

$m\dot{v}_z = qE$ The is accelaration in the z direction $a_z = \frac{qE}{m}$

$$z(t) = z_0 + v_{z0}t + \frac{1}{2}a_z t^2$$

The particle moves in a helix of const. radius around a line parallel to the z-axis with increasing pitch as the motion in the z direction accelerates