PHASE VELOCITY & DISPERSION

Let us consider a wave of single frequency given by

$$u(x,t) = A e^{i(kx-\omega t)}$$

If the argument of this exponential remains constant, the wave function $u(x,t)$ also remains constant. The quantity $kx-\omega t$ is defined as the **PHASE** of the wave represented by $u(x,t)$.

$$\phi \equiv kx - \omega t$$

If we move along the x-axis at such a velocity so that the phase at every point is the same, the wave pattern or form will remain unchanged with time. For ϕ to remain constant, we must have

$$\Delta\phi = d\phi = 0 \Rightarrow k\,dx - \omega\,dt = 0$$

From which we define the phase velocity V_p to be the velocity with which the wave pattern travels

$$V_p = \frac{dx}{dt} = \frac{\omega}{k} = V$$

That is, for a simple wave possessing a well-defined frequency, the phase velocity v_p is equal to the wave velocity v. In general, this is not true.

Usually the phase velocity is a function of frequency

$$v_p = v_p(k)$$

Such a medium, wherein $v_p = v_p(k)$, is called **DISPERSIVE**. In a dispersive medium the phase velocity is not equal to the wave velocity. (The best-known example of this phenomenon is the simple optical prism. The index of refraction depends upon the wavelength of the incident light).

In dispersive media the wave pattern is modified; it does not remain constant.

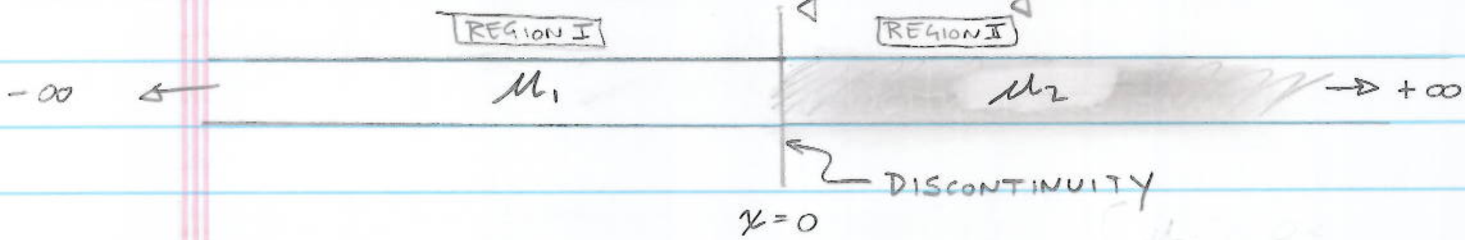
But even such a pattern will appear unchanged to an observer who is moving with a velocity V_g given by

$$V_g = \frac{d\omega(k)}{dk} \qquad \text{Here } \omega = \omega(k)$$

GROUP VELOCITY

Wave at a discontinuity: Energy Flow

Consider two semi-infinite strings of different linear mass densities joined together at $x = 0$



$$\mu = \begin{cases} \mu_1 & -\infty < x < 0 \\ \mu_2 & 0 < x < \infty \end{cases} \quad \leftarrow \text{linear mass density}$$

$$V = \begin{cases} V_1 & -\infty < x < 0 \\ V_2 & 0 < x < \infty \end{cases} \quad \leftarrow \text{wave velocity}$$

The incoming wave traveling from the left will be both reflected and transmitted at $x = 0$, where the mass discontinuity occurs.

<u>Goal</u>: Calculate the reflected and transmitted amplitudes.

<u>Region I</u> $\quad u_1(x,t) = u_{inc} + u_{refl}$

$$u_1(x,t) = A e^{i(\omega t - k_1 x)} + B e^{i(\omega t + k_1 x)}$$

<u>Region II</u> $\quad u_2(x,t) = u_{trans} = C e^{i(\omega t - k_2 x)}$

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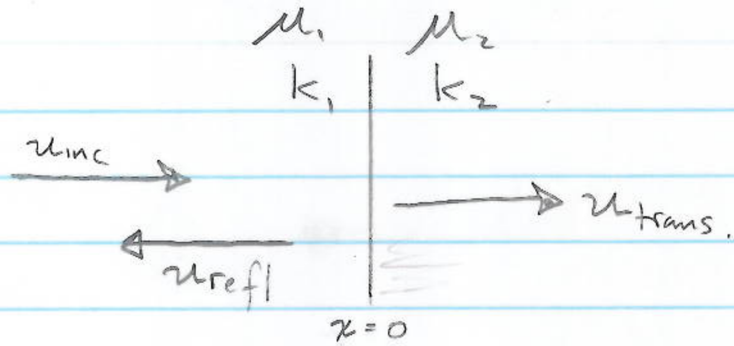

Boundary Conditions (CONTINUITY CONDITIONS)

① $\boxed{u_1 = u_2}$ (continuous across interface). This condition insures that there is no break in the string at the interface.

② $\boxed{\frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x}}$ (First derivative is continuous across the interface). This condition prevents a "kink" from occuring in the string @ $x=0$. This is to say the limit:

$$\lim_{\Delta x \to 0^+} \frac{\Delta u_1}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{\Delta u_2}{\Delta x}$$

If this derivative were not continuous, the second derivative of the wavefunction wrt x would be infinite and a finite force acting on the string element at this junction would produce an infinite acceleration.

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Imposing the first continuity condition yields

$$A + B = C$$

And from the second continuity condition:

$$-k_1 A + k_1 B = -k_2 C$$

The solution of these pair of equations gives

$$\frac{B}{A} = \frac{k_1 - k_2}{k_1 + k_2}, \quad \frac{C}{A} = \frac{2k_1}{k_1 + k_2}$$

Since $k = \frac{\omega}{v}$ and $v = \sqrt{T/\mu}$ we have

$$\frac{B}{A} = \frac{v_2 - v_1}{v_1 + v_2} = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}}$$

$$\frac{C}{A} = \frac{2v_2}{v_1 + v_2} = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1} + \sqrt{\mu_2}}.$$

It is clear that the ratio $\frac{C}{A}$ is always positive. The transmitted wave is therefore always in phase with the incident wave. If the second medium is less dense ($\mu_2 < \mu_1$), the ratio B/A will be positive and the reflected wave will be in phase with the incident wave. If, on the other hand, the second medium is denser than the first (i.e. $\mu_2 > \mu_1$), the ratio B/A will be negative and the reflected wave will be π out of phase with respect to the incident wave. [Can you make analogies to electromagnetic waves traveling from one medium having an index of refraction n_1 to another of n_2?]

The intensity (the energy flow per unit time per unit area) for any type of wave motion is proportional to the square of the amplitude.

We define the **REFLECTION COEFFICIENT**, $\mathbb{R}$, to be the fraction of the incident energy that is reflected back

$$\mathbb{R} \equiv \left(\frac{B}{A}\right)^2 = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 = \left(\frac{v_2 - v_1}{v_1 + v_2}\right)^2$$

No energy may be stored in the junction of the two strings, so the sum of the reflected and transmitted energies must equal the incident energy ($\mathbb{R} + \mathbb{T} = 1$)

Thus

$$\mathbb{T} = 1 - \mathbb{R} = \frac{4k_1k_2}{(k_1+k_2)^2} = \frac{4v_1v_2}{(v_1+v_2)^2}$$

We observe that $\mathbb{R}$ becomes larger as the difference between μ_1 and μ_2 (or v_1 and v_2) becomes larger, while the corresponding $\mathbb{T}$ becomes smaller.

Now let us calculate the rate of energy flow across the junction at $x=0$. This equals the time rate of change of the work done by the adjacent element of the string upon the element of string at $x=0$.

We found that this restoring force is equal to $-T\tan\theta$ and $\tan\theta = \frac{\partial u}{\partial x}$ (the minus sign comes from the fact that is restoring)

$$\frac{dW}{dt} = F_{restoring} \times V = \frac{dE}{dt}$$

$$\frac{dE}{dt} = \left(-T\frac{\partial u}{\partial x}\right)_{x=0} \times \left(\frac{\partial u}{\partial t}\right)_{x=0} \qquad (29)$$

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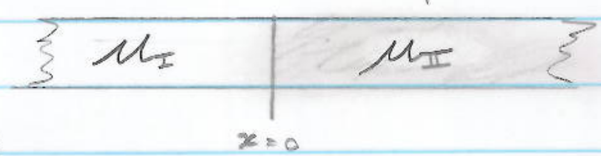
Let us use only the real portion of the waves, $\mathrm{Re}\{e^{i\theta}\} = \cos\theta$.

Again, A = Amplitude of incident wave
B = Amplitude of reflected wave
C = Amplitude of transmitted wave,

The energy flow for the left portion of the string at $x=0$:



$$u = u_{Inc} + u_{Refl.}$$

$$u_I = A\cos(k_1 x - \omega t) + B\cos(k_1 x + \omega t)$$

Substituting this into equ (29) gives:

$$\left(\frac{dE}{dt}\right)_I = \omega k_1 T (A^2 - B^2)\sin^2\omega t$$

Similarly, the flow of energy to the right of the junction is

$$u_{II} = C\cos(k_2 x - \omega t)$$

$$\Rightarrow \left(\frac{dE}{dt}\right)_{II} = \omega k_2 T C^2 \sin^2\omega t$$

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Since the average value of $\sin^2\omega t$ over one complete cycle is 1/2, we can write

$$\left\langle \left(\frac{dE}{dt}\right)_{I} \right\rangle = \frac{1}{2}\omega k_1 T A^2 - \frac{1}{2}\omega k_1 T B^2$$

mean rate at which energy is incident on the junction

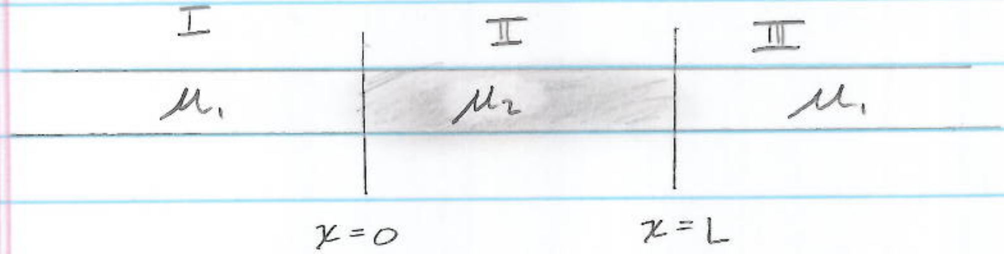
mean rate at which energy is reflected back.

Similarly.

$$\left\langle \left(\frac{dE}{dt}\right)_{II} \right\rangle = \frac{1}{2}\omega k_2 C^2$$

Mean rate at which energy is supplied to the junction from left to right

HINT



$$\mu = \begin{cases} \mu_1 & x<0;\ x>L \\ \mu_2 > \mu_1 & 0<x<L \end{cases}$$

Let the wave be incident from the left

$$u_{I} = Ae^{i(\omega t - k_1 x)} + Be^{i(\omega t + k_1 x)}$$

$$u_{II} = Ce^{i(\omega t - k_2 x)} + De^{i(\omega t + k_2 x)}$$

$$u_{III} = Ee^{i(\omega t - k_1 x)}$$

The reflected intensity is $I_R = I_0 \dfrac{|B|^2}{|A|^2}$

The transmitted intensity is $I_T = I_0 \dfrac{|E|^2}{|A|^2}$

You will need to apply the continuity conditions at $x=0$ and $x=L$ to relate A, B, C, D & E.