11-39

The transformational properties in orthogonal cartesian coordinates may be used here.

Let us start with a vector $\vec{L}$, the angular momentum, fixed in an inertial reference frame.

(*) $$L_k = \sum_l I_{kl}\,\omega_l$$

In a body coordinate system that is simply rotated with respect to the space coordinates, the angular momentum $\vec{L}'$ must have an analogous form

(1) $$L'_i = \sum_j I'_{ij}\,\omega'_j$$

Using the transformation properties of vectors, we may write the transformation of $\vec{L}$ and $\vec{\omega}$ as [See handout on coordinate transformations]

$$x_i = \sum_j \lambda^{\dagger}_{ij}\,x'_j = \sum_j \lambda_{ji}\,x'_j$$

Here λ_{ji} is an element of the transformation matrix

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix}$$

$$L_k = \sum_m \lambda_{mk} L'_m \qquad (2)$$

and

$$\omega_l = \sum_j \lambda_{jl} \omega'_j \qquad (3)$$

Substituting (2) and (3) into (*) gives

$$\sum_m \lambda_{mk} L'_m = \sum_l I_{kl} \sum_j \lambda_{jl} \omega'_j \qquad (4)$$

multiplying both sides of (4) by λ_{ik} and summing over k

$$\sum_m \left(\sum_k \lambda_{ik} \lambda_{mk} \right) L'_m = \sum_j \left(\sum_{k,l} \lambda_{ik} \lambda_{kl} I_{kl} \right) \omega'_j \qquad (5)$$

The left hand side may be written as

$$\sum_m \left(\sum_k \lambda_{ik} \lambda_{mk} \right) L'_m = \sum_m \delta_{im} L'_m = L'_i$$

That is

$$L'_i = \sum_j \left(\sum_{k,l} \lambda_{ik} \lambda_{jl} I_{kl} \right) \omega'_j \qquad (6)$$

Comparing (6) with (1) gives us the relation

$$I'_{ij} = \sum_{k,l} \lambda_{ik} \lambda_{jl} I_{kl}$$

Thus each element I_{kl} of inertia tensor $\mathbb{I}$ in a fixed coordinate system can be transformed into rotated (body) coordinates resulting in elements I'_{ij} of inertia tensor $\mathbb{I}'$

The preceding result may be written as

$$I'_{ij} = \sum_{k,l} \lambda_{ik} (I_{kl} \lambda_{lj})^t$$

But $I_{kl} = I^t_{kl}$ ← symmetric matrix

$$\Rightarrow \; I'_{ij} = \sum_{k,l} \lambda_{ik} I_{kl} \lambda^t_{lj}$$

where λ^t_{lj} are the elements of the transposed matrix $\boldsymbol{\lambda}^t$. Just as in matrix notation, we may write

$$\mathbb{I}' = \boldsymbol{\lambda} \mathbb{I} \boldsymbol{\lambda}^t$$

Since for othogonal transformations, $\boldsymbol{\lambda}^t = \boldsymbol{\lambda}^{-1}$, where $\boldsymbol{\lambda}^{-1}$ is the inverse matrix, we may write

$$\boxed{\mathbb{I}' = \boldsymbol{\lambda} \mathbb{I} \boldsymbol{\lambda}^{-1}}$$

This is the SIMILARITY TRANSFORMATION
($\mathbb{I}'$ IS SIMILAR to $\mathbb{I}$)

11-42

Let us rotate the coordinate system by an angle θ about the x_3-axis

$$\lambda = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\lambda^t = \lambda^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Observe that

$$\lambda\lambda^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Yes $\lambda\lambda^{-1} = \mathbb{1}$ as required.

Let us perform the similarity transformation on the diagonalized solid cube (pivot pt @ corner)

$$\{I\} = \begin{bmatrix} 1/6 & 0 & 0 \\ 0 & 11/12 & 0 \\ 0 & 0 & 11/12 \end{bmatrix} Mb^2$$

11-43.

$$\{I'\} = \lambda\{I\}\lambda^{-1} \qquad \begin{array}{l}\alpha = {}^{1}/_{6}\,mb^2 \\ \beta = {}^{11}/_{12}\,mb^2\end{array}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha\cos\theta & -\alpha\sin\theta & 0 \\ \beta\sin\theta & \beta\cos\theta & 0 \\ 0 & 0 & \beta \end{bmatrix}$$

$$\{I'\} = \begin{bmatrix} \alpha\cos^2\theta + \beta\sin^2\theta & \frac{1}{2}(\beta-\alpha)\sin 2\theta & 0 \\ \frac{1}{2}(\beta-\alpha)\sin 2\theta & \alpha\sin^2\theta + \beta\cos^2\theta & 0 \\ 0 & 0 & \beta \end{bmatrix}$$

We define the trace as:

$$\operatorname{tr}\{I\} \equiv \sum_k I_{kk}$$

$$\operatorname{tr}\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{bmatrix} = \alpha + \beta + \beta = \alpha + 2\beta \qquad (1)$$

$$\operatorname{tr}\{I'\} = (\alpha\cos^2\theta + \beta\sin^2\theta) + (\alpha\sin^2\theta + \beta\cos^2\theta) + \beta$$
$$= \alpha(\cos^2\theta + \sin^2\theta) + \beta(\cos^2\theta + \sin^2\theta) + \beta$$

$$= \alpha + 2\beta$$

Under similarity transformations, the trace is an INVARIANT QUANTITY

$$tr\{\mathbb{I}\} = tr\{\mathbb{I}'\}$$

You can prove this in 11-22.

The determinant of the elements of a tensor is an INVARIANT QUANTITY under a similarity transformation

$$\mathbb{I}' = \lambda \mathbb{I} \lambda^{-1}$$

$$|\mathbb{I}'| = |\lambda \mathbb{I} \lambda^{-1}| = |\lambda| \times |\mathbb{I}| \times |\lambda^{-1}|$$
$$= |\mathbb{I}| \times |\lambda| \times |\lambda^{-1}| = |\mathbb{I}| \times |\lambda \lambda^{-1}|$$
$$= |\mathbb{I}| \times |\mathbb{1}| = |\mathbb{I}'|$$

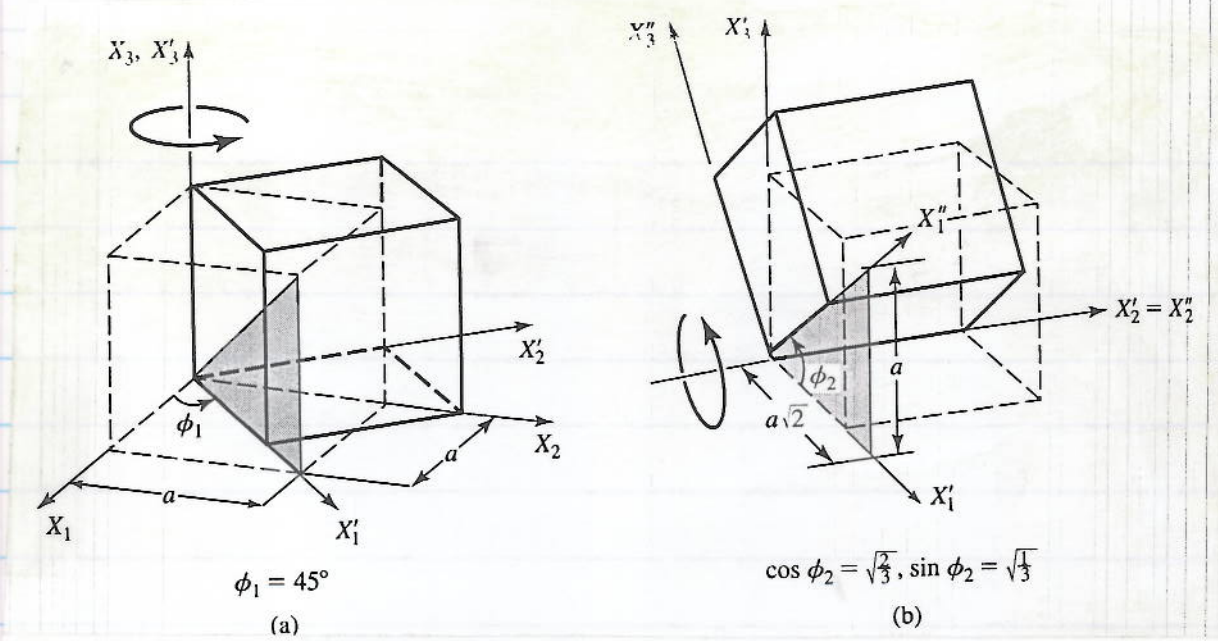
→ For your HW I want you to verify this with the cube pivoted at the corner (DIAGONALIZED and axes along edges.)

i.e.

$$\{\mathbb{I}\} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{bmatrix} mb^2 \text{ and}$$

$$\{\mathbb{I}'\} = \begin{bmatrix} 1/6 & 0 & 0 \\ 0 & 11/12 & 0 \\ 0 & 0 & 11/12 \end{bmatrix} Mb^2$$

11-45



We seek to diagnalize the inertia tensor of a uniform solid cube pivoted at one corner
The initial axes are along the side of the cube
We have:

$$\{I\} = \begin{bmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{bmatrix} ma^2$$

From our earlier analysis we found that for the diagnalized matrix, one of the principal axes is along the BODY DIAGONAL

Let us rotate the axis through an angle of $\frac{\pi}{4}$ about the x_3-axis

$$\lambda_1 = \begin{bmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} & 0 \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We now rotate the body axis through an angle of $\phi_2 = \cos^{-1}\left(\sqrt{\frac{2}{3}}\right)$ about the x_2' axis. Observe that the x_1'' is now along the body diagonal.

$$\lambda_2 = \begin{bmatrix} \cos\phi_2 & 0 & \sin\phi_2 \\ 0 & 1 & 0 \\ -\sin\phi_2 & 0 & \cos\phi_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2/3} & 0 & \sqrt{1/3} \\ 0 & 1 & 0 \\ -\sqrt{1/3} & 0 & \sqrt{2/3} \end{bmatrix}$$

The complete rotation matrix is

$$\lambda = \lambda_2 \lambda_1 \quad \leftarrow \text{Note the order!}$$

$$\lambda = \begin{bmatrix} \sqrt{2/3} & 0 & \sqrt{1/3} \\ 0 & 1 & 0 \\ -\sqrt{1/3} & 0 & \sqrt{2/3} \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\lambda = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ -\sqrt{3}/2 & \sqrt{3}/2 & 0 \\ -\sqrt{2}/2 & -\sqrt{2}/2 & \sqrt{2} \end{bmatrix}$$

$$\lambda^{-1} = \lambda^t = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -\sqrt{3}/2 & -\sqrt{2}/2 \\ 1 & \sqrt{3}/2 & -\sqrt{2}/2 \\ 1 & 0 & \sqrt{2} \end{bmatrix} \leftarrow$$

Note that for orthogonal transformations $\lambda^{-1} = \lambda^t$!!

EASY TO INVERT MATRIX!

11-47

Let $\beta = ma^2$. Performing the similarity transformation

$$\{I'\} = \frac{\beta}{3}\begin{bmatrix} 1 & 1 & 1 \\ -\sqrt{3}/2 & \sqrt{3}/2 & 0 \\ -\sqrt{2}/2 & -\sqrt{2}/2 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{bmatrix}\begin{bmatrix} 1 & -\sqrt{3}/2 & -\sqrt{2}/2 \\ 1 & \sqrt{3}/2 & -\sqrt{2}/2 \\ 1 & 0 & \sqrt{2} \end{bmatrix}$$

$$\{I'\} = \begin{bmatrix} \frac{1}{6}\beta & 0 & 0 \\ 0 & \frac{11}{12}\beta & 0 \\ 0 & 0 & \frac{11}{12}\beta \end{bmatrix}$$

Et voilà, the matrix is diagonalized!

Eulerian ANGLES

We are interested in evaluating a matrix that will enable us to transform from one coordinate system to another. Let us say we wish to rotate from coordinates $\vec{x}'$ of a fixed coordinate system to $\vec{x}$ of the body coordinate system

$\vec{x}'$: fixed or inertial coordinate system
$\vec{x}$: body coordinate system

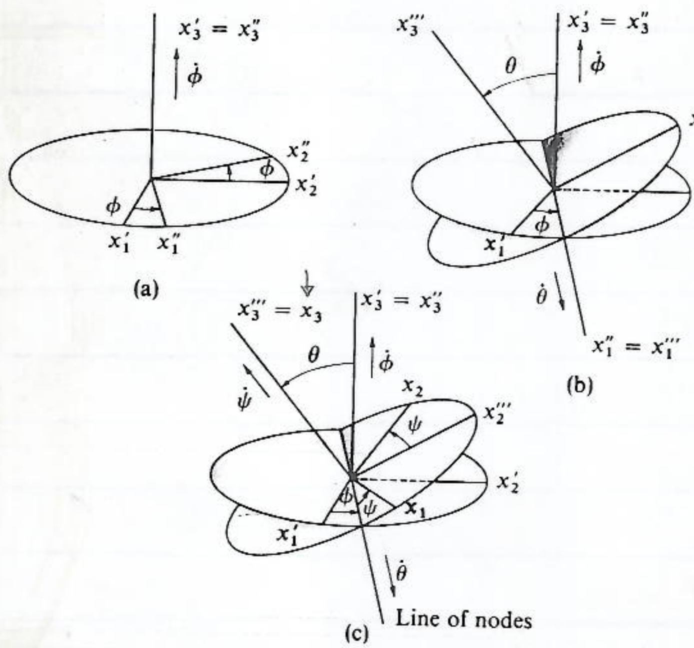


FIGURE 11-7

Sequence of three angular rotations employed in going from the system $\vec{x}'$ to the system $\vec{x}$

The transformation may be represented by the matrix equation

$$\vec{x} = \lambda \vec{x}'$$

The rotation matrix λ COMPLETELY specifies the relative orientation of the two systems. There are a host of three INDEPENDENT angles which can be used in the ROTATION MATRIX. The most common and convenient angles used are the

Eulerian Angles

represented as ϕ, θ, and ψ.

To go from an x_i' to an x_i, one must perform the following sequence of rotations

counterclockwise

① The first rotation is through an angle ϕ about the x_3' axis and is in the $x_1'-x_2'$ plane as shown in the figure above. The transformation matrix for this rotation in the $x_1'-x_2'$ plane is:

$$\lambda_\phi = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The angle ϕ is called the Precession Angle

$$\vec{x}'' = \lambda_\phi \vec{x}'$$

② The second rotation is counterclockwise through an angle θ about the x_1''-axis, and in the x_2''-x_3'' plane, transforming the axes $x_i'' \to x_i'''$ as shown in the figure above. The transformation matrix for this rotation in the x_2''-x_3'' plane is

$$\lambda_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

The angle θ is called the NUTATION ANGLE

$$\vec{x}''' = \lambda_\theta \vec{x}''$$

③ The third rotation is counterclockwise through an angle ψ about the x_3'''-axis and is in the x_1'''-x_2''' plane, transforming the axes $x_i''' \to x_i$ as is shown in the figure above. The transformation matrix for this rotation is

$$\lambda_\psi = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The angle ψ is called the BODY ANGLE

$$\vec{x} = \lambda_\psi \vec{x}'''$$

The line common to the planes containing the x_1- and x_2-axes and the x'_1- and x'_2-axes is called the LINE of NODES. The complete transformation from the x'_i system to the x_i system is given by

$$\vec{x} = \lambda_\psi \vec{x}''' = \lambda_\psi (\lambda_\theta \vec{x}'') = \lambda_\psi \lambda_\theta \vec{x}''$$

$$= \lambda_\psi \lambda_\theta (\lambda_\phi \vec{x}') = \lambda_\psi \lambda_\theta \lambda_\phi \vec{x}'$$

$$\vec{x} = \lambda \vec{x}'$$

where the rotation matrix is

$$\lambda = \lambda_\psi \lambda_\theta \lambda_\phi$$

$$\lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And the components of this matrix are

$\lambda_{11} = \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi$
$\lambda_{21} = -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi$
$\lambda_{31} = \sin\theta\sin\phi$

$\lambda_{12} = \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi$
$\lambda_{22} = -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi$
$\lambda_{32} = -\sin\theta\cos\phi$

$\lambda_{13} = \sin\psi\sin\theta$
$\lambda_{23} = \cos\psi\sin\theta$
$\lambda_{33} = \cos\theta$

All infinitesimal rotations can be represented by vector notation. This enables us to represent the three time derivatives of rotation ($\dot\phi$, $\dot\theta$, and $\dot\psi$) as the components of an angular velocity vector

$$\vec{\omega} = (\omega_\phi, \omega_\theta, \omega_\psi)$$

The three components of $\vec{\omega}$ are not all either along the fixed axes or the body axes

$\omega_\phi = \dot{\phi}$ is directed along the x_3' (fixed) axis

$\omega_\theta = \dot{\theta}$ is directed along the line of nodes

$\omega_\psi = \dot{\psi}$ is directed along the x_3 (body) axis

It is not convenient to use these components to describe the motion of a rigid body. The rigid body equations of motion are best described in the BODY COORDINATE SYSTEM. We therefore must calculate the angular velocity vector $\vec{\omega}$ in the body coordinate system. And to do this, we must decompose $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$ along the body axes: (see figure)

$$\dot{\phi}_1 = \dot{\phi}\sin\theta\sin\psi \qquad \dot{\phi}_2 = \dot{\phi}\sin\theta\cos\psi \qquad \dot{\phi}_3 = \dot{\phi}\cos\theta$$

$$\dot{\theta}_1 = \dot{\theta}\cos\psi \qquad \dot{\theta}_2 = -\dot{\theta}\sin\psi \qquad \dot{\theta}_3 = 0$$

← along line of nodes.

$$\dot{\psi}_1 = 0 \qquad \dot{\psi}_2 = 0 \qquad \dot{\psi}_3 = \dot{\psi}$$

It is important to emphasize that the angular displacements and other rotational quantities can be represented as vector iff these rotations are infinitesimally small; they then obey the law of vector addition.

Collecting the components of $\vec{\omega}$ gives us Euler's geometric equations

$$\boxed{\begin{aligned} \omega_1 &= \dot{\phi}_1 + \dot{\theta}_1 + \dot{\psi}_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_2 &= \dot{\phi}_2 + \dot{\theta}_2 + \dot{\psi}_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_3 &= \dot{\phi}_3 + \dot{\theta}_3 + \dot{\psi}_3 = \dot{\phi}\cos\theta + \dot{\psi} \end{aligned}}$$

Euler's Equations of motion for a RIGID BODY

The rotational motion of a body is described by

$$\vec{\tau} = \frac{d\vec{L}}{dt}, \quad \text{where} \quad \vec{L} = I\vec{\omega} \quad \left(\text{c.f. NII } \vec{F} = \frac{d\vec{p}}{dt}\right)$$

Let us now proceed to obtain Euler's equations of motion for a rigid body in a force field. The above equation describes the motion of a rigid body AS VIEWED FROM A FIXED, INERTIAL, or LABORATORY COORDINATE SYSTEM.

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\overleftrightarrow{I}\cdot\vec{\omega}) = \vec{\tau}$$

This double-headed arrow indicates that I is a second rank tensor.

Please observe that the moment of inertia tensor $\overleftrightarrow{I}$ changes as the body rotates.

We have from equation (10.6) in section 10.1 Rotating coordinate systems

$$\left(\frac{d\vec{r}}{dt}\right)_{fixed} = \left(\frac{d\vec{r}}{dt}\right)_{rotating} + \vec{\omega}\times\vec{r}$$

$$\Rightarrow \left(\frac{d\vec{L}}{dt}\right)_{fixed} = \left(\frac{d\vec{L}}{dt}\right)_{rotating} + \vec{\omega}\times\vec{L} = \vec{\tau}$$

$$= \left(\frac{d}{dt}(\overleftrightarrow{I}\cdot\vec{\omega})\right)_{rotating} + \vec{\omega}\times(\overleftrightarrow{I}\cdot\vec{\omega}) = \vec{\tau}$$

$$\frac{d\vec{L}}{dt} = \overleftrightarrow{I} \cdot \frac{d\vec{\omega}}{dt}\Big|_{\text{rotating}} + \frac{d\overleftrightarrow{I}}{dt}\Big|_{\text{rotating}} \cdot \vec{\omega} + \vec{\omega} \times (\overleftrightarrow{I} \cdot \vec{\omega}) = \vec{\tau}$$

But in the rotating frame

$$\frac{d\overleftrightarrow{I}}{dt}\Big|_{\text{rotating}} = 0$$

We also note that the angular acceleration $\dot{\vec{\omega}}$ is the same in both the rotating and fixed frames.

$$\left(\frac{d\vec{\omega}}{dt}\right)_{\text{fixed}} = \left(\frac{d\vec{\omega}}{dt}\right)_{\text{rotating}} + \underbrace{\vec{\omega} \times \vec{\omega}}_{\to 0}$$

$$\frac{d\vec{L}}{dt} = \overleftrightarrow{I} \cdot \frac{d\vec{\omega}}{dt} + \vec{\omega} \times (\overleftrightarrow{I} \cdot \vec{\omega}) = \vec{\tau} \qquad (*)$$

For the sake of convenience let us choose the body axes to be the principal axes so that

$$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \omega_1 \hat{x}_1 \\ \omega_2 \hat{x}_2 \\ \omega_3 \hat{x}_3 \end{bmatrix}$$

$$\Rightarrow \vec{L} = I_1 \omega_1 \hat{x}_1 + I_2 \omega_2 \hat{x}_2 + I_3 \omega_3 \hat{x}_3$$

That is

$$L_1 = I_1\omega_1 \qquad L_2 = I_2\omega_2 \qquad L_3 = I_3\omega_3$$

$$\vec{\omega} \times (\overleftrightarrow{I} \cdot \vec{\omega}) = \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ I_1\omega_1 & I_2\omega_2 & I_3\omega_3 \end{vmatrix}$$

$$= (I_3 - I_2)\,\omega_2\omega_3\,\hat{x}_1 + (I_1 - I_3)\,\omega_1\omega_3\,\hat{x}_2 + (I_2 - I_1)\,\omega_1\omega_2\,\hat{x}_3$$

Equ (*) becomes in component form:

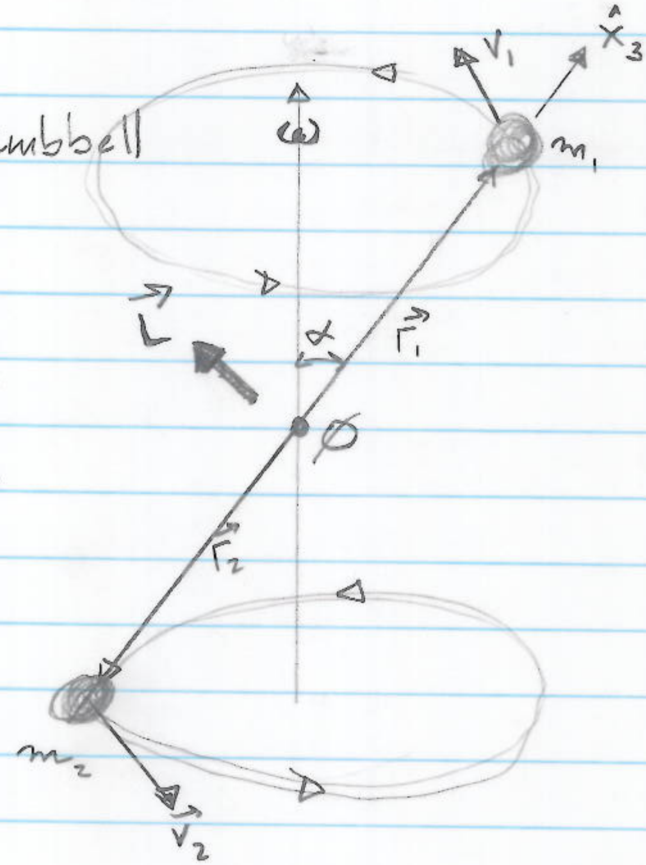
$$\boxed{\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\,\omega_2\omega_3 &= \tau_1 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\,\omega_1\omega_3 &= \tau_2 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\,\omega_1\omega_2 &= \tau_3 \end{aligned}}$$

These relations are known as Euler's Dynamical Equations for the motion of a rigid body in a force field

For force-free motion (in the absence of torque) the Euler Dynamical Equations become

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = 0$$
$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = 0$$
$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = 0$$

Let us consider the dumbbell



Let $|\vec{r}_1| = |\vec{r}_2| = b$.
Let the body coordinate system be pivoted at ϕ (i.e. its origin is at ϕ) and the symmetry axis x_3 be along the weightless shaft toward m_1

$$\vec{L} = \sum_{i=1}^{2} m_i \vec{r}_i \times \vec{V}_i$$

We observe the $\vec{L} \perp \hat{x}_3$ ($\vec{L} \perp$ shaft) and $\vec{L}$ traces out a cone about $\vec{\omega}$. Let $\hat{x}_2$ be along $\vec{L}$

$$\vec{L} = L_2\hat{x}_2$$

11-58

If α is the angle between $\vec{\omega}$ and the shaft, the components of $\vec{\omega}$ are

$$\omega_1 = 0$$
$$\omega_2 = \omega \sin\alpha$$
$$\omega_3 = \omega \cos\alpha$$

The principle axes are in the direction of $\hat{x}_1$, $\hat{x}_2$, and $\hat{x}_3$

$$I_{xx} = \sum_{i=1}^{2} m_i (x_{i2}^2 + \cancelto{0}{x_{i3}^2}) = (m_1 + m_2) b^2 = I_1$$

$$I_{xy} = I_{yx} = 0$$

$$I_{yy} = \sum_{i=1}^{2} m_i (\cancelto{0}{x_{i1}^2} + x_{i3}^2) = (m_1 + m_2) b^2 = I_2$$

$$I_{yz} = I_{zy} = 0$$

$$I_{zz} = \sum_{i=1}^{2} m_i (x_{i1}^2 + x_{i2}^2) = 0$$

or
$$I_1 = (m_1 + m_2) b^2$$
$$I_2 = (m_1 + m_2) b^2$$
$$I_3 = 0$$

$$L_1 = I_1 \omega_1 = 0$$
$$L_2 = I_2 \omega_2 = (m_1 + m_2) b^2 \omega \sin\alpha$$
$$L_3 = I_3 \omega_3 = 0$$

Making use of Euler's Dynamical Equations and setting $|\vec{\omega}| = \text{const}$, i.e. $\dot{\omega} = 0$

$$\tau_1 = -(m_1+m_2)b^2\omega^2 \sin\alpha\cos\alpha$$
$$\tau_2 = \tau_3 = 0$$

This is the torque required to maintain this motion if $\dot{\omega}=0$ is directed along the x_1-axis

FORCE-FREE MOTION of a SYMMETRIC TOP

Let us solve Euler's Dynamical Equations for the special case in which $\vec{\tau}=0$. We shall investigate the case in which two of the principal axes of inertia are the same.

$$I_1 = I_2 \neq I_3$$

Let $I_1 = I_2 = I_{12}$. The Euler Dynamical Equations reduce to

(1a) $$I_{12}\dot{\omega}_1 + (I_3 - I_{12})\omega_2\omega_3 = 0$$

(1b) $$I_{12}\dot{\omega}_2 + (I_{12} - I_3)\omega_1\omega_3 = 0$$

(1c) $$I_3\dot{\omega}_3 = 0$$

11-60

Let's make two points clear.

(1) The motion is force free. The center of mass of the body is either at rest or is moving with constant velocity. WLOG, we shall assume that the center of mass is at rest and we shall set the origin in the fixed or laboratory frame at the center of mass

(2) The angular velocity $\vec{\omega}$ does not lie along one of the principal axes. (otherwise the problem would become trivial)

From equ. (1c), since $I_3 \neq 0$

$$\dot{\omega}_3 = 0$$

which upon integration yields

$$\omega_3(t) = \text{constant} \tag{2}$$

This equation states that for any rigid body rotating with angular velocity $\vec{\omega}$, the component of angular velocity along the symmetry axis, ω_3, remains constant. (If $\vec{\omega}$ were along the x_3-axis, the principal axis, the entire angular velocity would remain constant.)

Equations (1a) and (1b) may be written as

$$\dot{\omega}_1 + \frac{I_3 - I_{12}}{I_{12}} \omega_3 \omega_2 = 0$$

$$\dot{\omega}_2 - \frac{I_3 - I_{12}}{I_{12}} \omega_3 \omega_1 = 0$$

Let us define Ω_B B = BODY

$$\Omega_B \equiv \frac{I_3 - I_{12}}{I_{12}} \omega_3$$

This implies

(3a) $\dot{\omega}_1 + \Omega_B \omega_2 = 0$

(3b) $\dot{\omega}_2 - \Omega_B \omega_1 = 0$

We shall multiply (3b) by i and add it to (3a)

(4) $(\dot{\omega}_1 + i\dot{\omega}_2) - i\Omega_B(\omega_1 + i\omega_2) = 0$

We shall define

$$\eta \equiv \omega_1 + i\omega_2 \qquad (5a)$$

then

$$\dot{\eta} = \dot{\omega}_1 + i\dot{\omega}_2 \qquad (5b)$$

Substituting (5a) and (5b) into (4)

$$\dot{\eta} - i\Omega\eta = 0 \qquad (6)$$

Assuming the phase angle $\delta = 0$ @ $t = 0$, the solution of (6) is

Here A IS NOT COMPLEX

$$\eta = \omega_1 + i\omega_2 = Ae^{i\Omega t} = A\cos\Omega_B t + iA\sin\Omega_B t$$

comparing sides:

$$\left.\begin{aligned} &(7a) \quad \omega_1(t) = A\cos\Omega_B t \\ &(7b) \quad \omega_2(t) = A\sin\Omega_B t \end{aligned}\right\} \omega_1^2 + \omega_2^2 = A^2$$

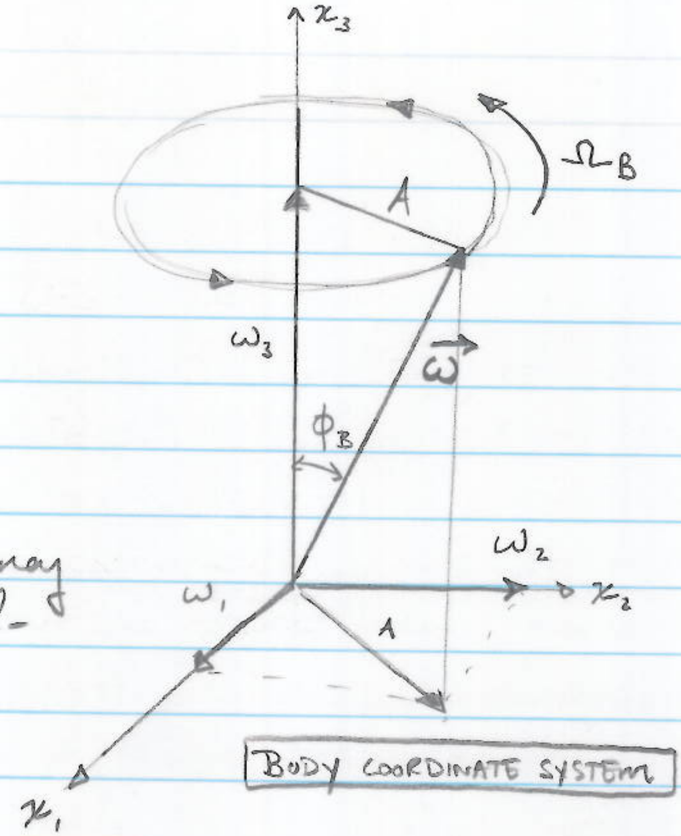
The magnitude of $\vec{\omega}$ is also constant since ω_3 is constant

$$|\vec{\omega}| = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \sqrt{A^2 + \omega_3^2} = \text{constant}$$

Equations (7a) and (7b) are parametric equations of a circle, and ω_1 & ω_2 are the components of $\vec{\omega}$ in the x_1-x_2 body plane. Thus, the components ω_1 and ω_2 trace out a circle with time in the x_1-x_2 plane. This implies that the angular velocity vector $\vec{\omega}$ precesses about the X_3-axis IN THE BODY COORDINATE SYSTEM with a constant angular frequency Ω_B

11-63

To an observer in the body coordinate system the angular velocity vector $\vec{\omega}$ precesses in a cone about the x_3-axis, with a constant angular frequency Ω_B. Here ϕ_B is the half-angle of the BODY CONE



In the body reference frame, the half angle of the BODY CONE is

$$\tan\phi_B = \frac{(\omega_1^2+\omega_2^2)^{1/2}}{\omega_3} = \frac{A}{\omega_3}$$

Recall that we have been considering the force-free motion of a rigid body. As viewed from the inertial system (i.e. the lab frame), there are two constants of the motion

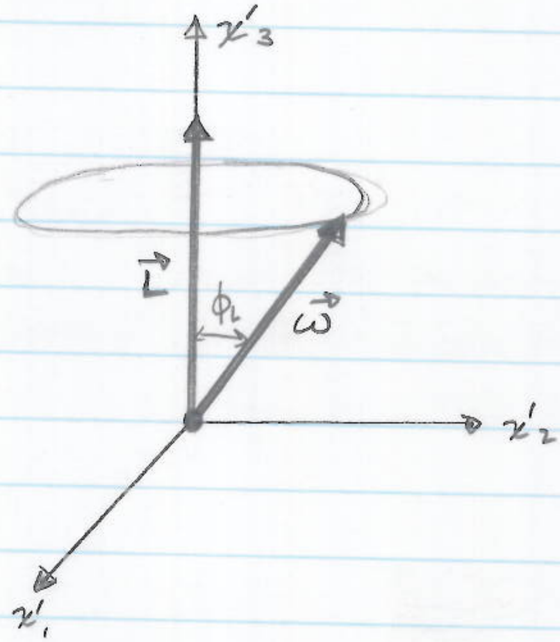
NO TORQUE

1) Angular momentum $\vec{L}(t) = \text{constant}$

2) Kinetic Energy Since the center of mass is fixed, the kinetic energy is purely rotational

$$T_{rot} = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \text{constant}$$

Since $\vec{L}$ is constant, T_{rot} will be constant iff $\vec{\omega}$ moves in such a way that its projection on the angular momentum vector $\vec{L}$ or the x_3'-axis is constant. As shown in the figure, the angle ϕ_L between $\vec{\omega}$ and $\vec{L}$ is given by



$$\cos\phi_L = \frac{\vec{\omega}\cdot\vec{L}}{\omega L}$$

$$\cos\phi_L = \frac{2T_{rot}}{\omega L} = \text{const}$$

Angle ϕ_L remains constant and is the half-angle of the LABORATORY or SPACE CONE.

This cone is the result of the precession of $\vec{\omega}$ about the constant angular momentum $\vec{L}$ as viewed from the inertia or Laboratory frame. In this case, $\vec{L}$, $\vec{\omega}$, and $\hat{x}_3'$ all lie in a plane, and since $\vec{L}$ is designated to be along the $\hat{x}_3'$-axis, $\vec{\omega}$ will precess about the x_3'-axis when viewed in the lab frame. On the other hand, when viewed from the body frame $\vec{\omega}$ precesses about the x_3 (BODY) axis. The situation is shown below; we have one cone rolling on another: $\vec{\omega}$ precesses around the x_3-axis in the body system and around the x_3' in the lab frame.

11-65

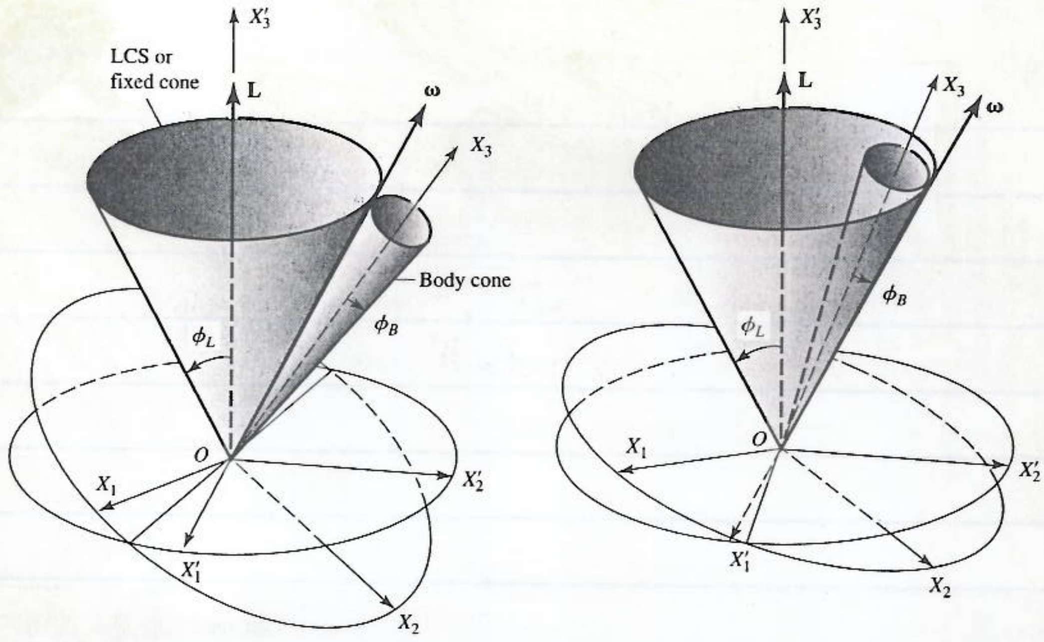


Figure 13.11 Body cone rolling around a LCS cone without slipping. Depending on the values of I_{12} and I_3, the body cone may roll (a) outside or (b) inside the LCS cone.

Now let us make use of the analytical solutions for the Euler angles as a function of time. For a symmetric body precessing uniformly in the absence of torques

$$I_1 = I_2 \neq I_3$$

$$\omega_1 = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$\omega_2 = \dot{\varphi}\sin\theta\cos\psi + \dot{\theta}\sin\psi$$
$$\omega_3 = \dot{\varphi}\cos\theta + \dot{\psi}$$

Let the $\hat{x}_1$-axis be taken along the line of nodes, i.e. $\psi = 0$.

$$\omega_1 = \dot{\theta}$$
$$\omega_2 = \dot{\varphi}\sin\theta \qquad \text{or BODY FRAME}$$
$$\omega_3 = \dot{\varphi}\cos\theta + \dot{\psi}$$

11-66

Let $\vec{L} \parallel \hat{x}_3'$-axis and constant (with the $\hat{x}_1$-axis coinciding with the line of nodes).

IN THE LAB FRAME.

$$L_1 = I_{12}\omega_1 = I_{12}\dot{\theta}$$
$$L_2 = I_{12}\omega_2 = I_{12}\dot{\varphi}\sin\theta$$
$$L_3 = I_3\omega_3 = I_3(\dot{\varphi}\cos\theta + \dot{\psi})$$

Because $\hat{x}_1' \perp \hat{x}_3$

$$L_1 = 0 \qquad L_2 = L\sin\theta \qquad L_3 = L\cos\theta$$

comparing with the above

$$\dot{\theta} = 0 \Rightarrow \boxed{\theta = \text{const}}$$

The angle between the axis of the symmetrical body and $\vec{L}$ is constant.

$$I_{12}\dot{\varphi}\sin\theta = L\sin\theta$$
$$\Rightarrow \boxed{\dot{\varphi} = \frac{L}{I_{12}}}$$

The angular velocity of precession

$$\boxed{\varphi = \frac{L}{I_{12}}t + K}$$

$$I_3(\dot{\varphi}\cos\theta + \dot{\psi}) = L\cos\theta$$
$$\dot{\psi} = \frac{L\cos\theta}{I_3} - \dot{\varphi}\cos\theta = L\cos\theta\left[\frac{1}{I_3} - \frac{1}{I_{12}}\right]$$

$$\boxed{\psi = L\cos\theta\left(\frac{I_{12} - I_3}{I_{12}I_3}\right)t + K'}$$