Dynamics of Rigid Bodies

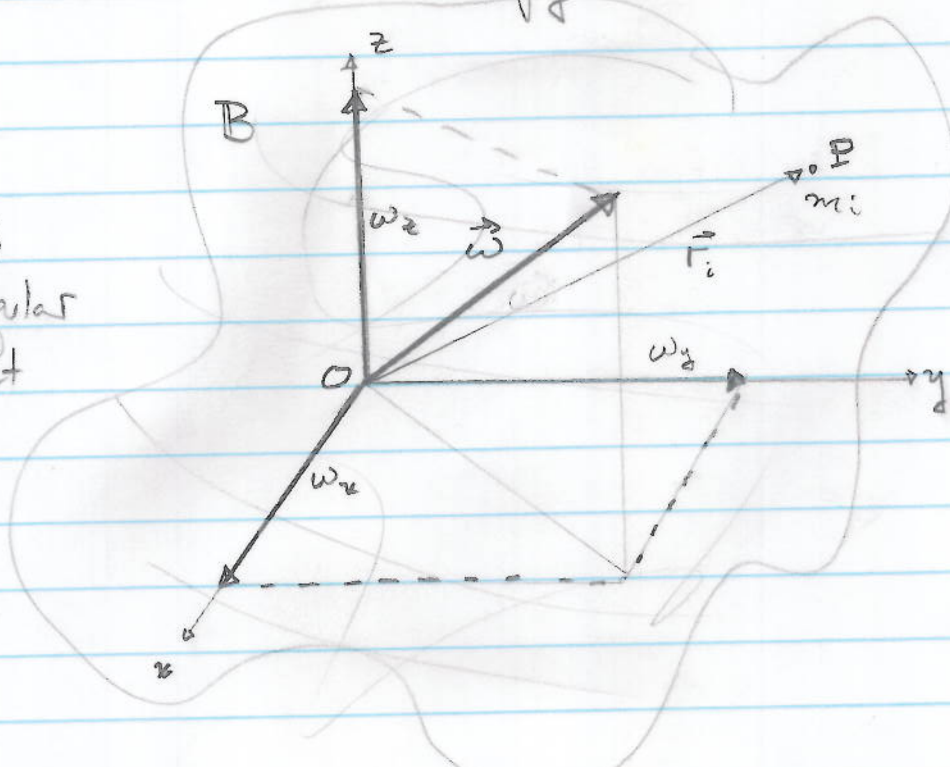
Rigid Body: A collection of discrete point particles for which the distance between any pair of particles is constrained to remain constant with time

To specify the position of the body, six coordinates are needed:

3 to specify the center of mass
3 to specify the body coordinate axis wrt the inertial (or fixed) coordinate axes } usually Eulerian Angles

Angular Momentum & Kinetic Energy

Rigid Body B rotating with angular velocity $\vec{\omega}$ about an axis passing through a single fixed point O



The instantaneous translational velocity $\vec{v}_i$ of particle P having mass m_i at a distance $\vec{r}_i$ from the origin O is

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i$$

has components $(\omega_x \ \omega_y \ \omega_z)$

The angular momentum $\vec{L}$ relative to the origin O, for a system of particles m_i can be defined:

$$\vec{L} = \sum_{i=1}^{n} m_i \vec{r}_i \times \vec{v}_i$$

$$= \sum_{i=1}^{n} m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) \qquad (1)$$

Using the identity for the triple cross product

$$\vec{A} \times (\vec{B} \times \vec{A}) = A^2 \vec{B} - \vec{A}(\vec{A} \cdot \vec{B})$$

we may write

$$\vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = r_i^2 \vec{\omega} - \vec{r}_i (\vec{r}_i \cdot \vec{\omega})$$

$$(2) \qquad = (x_i^2 + y_i^2 + z_i^2)(\hat{i}\omega_x + \hat{j}\omega_y + \hat{k}\omega_z)$$

$$- (\hat{i}x_i + \hat{j}y_i + \hat{k}z_i)(x_i\omega_x + y_i\omega_y + z_i\omega_z)$$

Combining (1) and (2) and rearranging

$$\vec{L} = \hat{i}L_x + \hat{j}L_y + \hat{k}L_z \qquad (3)$$

$$= \hat{i}\left[\omega_x \sum_{i=1}^{n} m_i(y_i^2+z_i^2) - \omega_y \sum_{i=1}^{n} m_i x_i y_i - \omega_z \sum_{i=1}^{n} m_i x_i z_i\right]$$

$$+ \hat{j}\left[-\omega_x \sum_{i=1}^{n} m_i y_i x_i + \omega_y \sum_{i=1}^{n} m_i(x_i^2+z_i^2) - \omega_z \sum_{i=1}^{n} m_i y_i z_i\right]$$

$$+ \hat{k}\left[-\omega_x \sum_{i=1}^{n} m_i z_i x_i - \omega_y \sum_{i=1}^{n} m_i z_i y_i + \omega_z \sum m_i(x_i^2+y_i^2)\right]$$

We may obtain the same result by making use of matrix expansion

$$\vec{r}_i \times (\omega \times \vec{r}_i) = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_i & y_i & z_i \\ (\omega_y z_i - \omega_z y_i) & (\omega_z x_i - \omega_x z_i) & (\omega_x y_i - \omega_y x_i) \end{bmatrix}$$

We may write (3) as.

$$\vec{L} = \hat{i}L_x + \hat{j}L_y + \hat{k}L_z \qquad (4)$$

$$= \hat{i}\left[\omega_x I_{xx} - \omega_y I_{xy} - \omega_z I_{xz}\right]$$

$$+ \hat{j}\left[-\omega_x I_{yx} + \omega_y I_{yy} - \omega_z I_{yz}\right]$$

$$+ \hat{k}\left[-\omega_x I_{zx} - \omega_y I_{zy} + \omega_z I_{zz}\right]$$

We have:

$$I_{xy} = I_{yx} = \sum m_i x_i y_i \quad \leftarrow \text{xy product of inertia}$$

$$I_{yz} = I_{zy} = \sum m_i y_i z_i \quad \leftarrow \text{yz product of inertia}$$

$$I_{zx} = I_{xz} = \sum m_i z_i x_i \quad \leftarrow \text{zx product of inertia}$$

It is clear from (4) that $\vec{L}$ is not necessarily always in the same direction as the INSTANTANEOUS AXIS of ROTATION. ← $\boxed{\text{if } \vec{L} \nparallel \vec{\omega} \text{: wobble}}$

That is $\vec{L}$ need not be in the same direction as $\vec{\omega}$

For example, if the z-axis is the direction of rotation $[\vec{\omega} = (0,0,\omega)]$ and from equation (4)

$$L_x = -I_{xz}\,\omega$$

$$L_y = -I_{yz}\,\omega$$

$$L_z = +I_{zz}\,\omega$$

← $\vec{L}$ and $\vec{\omega}$ are not in the same direction.

That is, $\vec{L}$ has a component $L_z = I_{zz}\,\omega$ in the direction of rotation, but has also two components that are $\perp$ to the direction of rotation.

The quantities I_{xx}, I_{yy}, and I_{zz} involve the sums of the squares of the coordinates and are called:

MOMENTS of INERTIA

of that body about the coordinate axis

That is, the summation is taken from $i=1$ to n

$$I_{xx} = \sum m_i (y_i^2 + z_i^2) = \sum m_i (r_i^2 - x_i^2)$$ moment of inertia about the x-axis

$$I_{yy} = \sum m_i (x_i^2 + z_i^2) = \sum m_i (r_i^2 - y_i^2)$$ moment of inertia about the y-axis

$$I_{zz} = \sum m_i (x_i^2 + y_i^2) = \sum m_i (r_i^2 - z_i^2)$$ moment of inertia about the z-axis

The quantities I_{xy}, I_{xz}, ... involve the sums of the products of the coordinates and are called the products

PRODUCTS of INERTIA

of that body about the coordinate axis

The components of $\vec{L}$ given by (4) may be written in compact form:

$$L_k = \sum_{l=1}^{3} \omega_l I_{kl}$$ ← k = 1,2,3 and l = 1,2,3
i.e. we have replaced
x, y, z by 1, 2, 3

We are now in a position to derive a general expression for the rotational kinetic energy of a body. In simple cases, the axis of rotation always remains normal to a fixed plane (i.e. wobble – tire not balanced)

This need not be the case, however. Let us calculate the kinetic energy of a rigid body that is rotating about an axis passing through a fixed point with angular velocity $\vec{\omega}$

$$\vec{V}_i = \vec{\omega} \times \vec{r}_i$$

The Kinetic Energy of the whole body is then

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i V_i^2 = \frac{1}{2} \sum_{i=1}^{n} m_i \vec{V}_i \cdot \vec{V}_i = \frac{1}{2} \sum_{i=1}^{n} \left[(\vec{\omega} \times \vec{r}_i) \cdot (m_i \vec{V}_i) \right]$$

But in a triple scalar product, the dot and cross may be interchanged; that is

$$(\vec{A} \times \vec{B}) \cdot \vec{C} = \vec{A} \cdot (\vec{B} \times \vec{C})$$

$$(\vec{\omega} \times \vec{r}_i) \cdot m_i \vec{v}_i = \vec{\omega} \cdot (\vec{r}_i \times m_i \vec{v}_i)$$

The kinetic energy T can then be written

$$T = \frac{1}{2} \sum_{i=1}^{n} \left[\vec{\omega} \cdot (\vec{r}_i \times m_i \vec{v}_i) \right]$$

Since $\vec{\omega}$ is the same for all particles

$$T = \frac{1}{2} \vec{\omega} \cdot \underbrace{\left[\sum_{i=1}^{n} (\vec{r}_i \times m_i \vec{v}) \right]}_{\vec{L}}$$

cf (1). (Definition)

$$\boxed{T_{rot} = \frac{1}{2} \vec{\omega} \cdot \vec{L}}$$

← observe that T is a scalar

Note that this expression is analogous to the expression for translational kinetic energy

$$\boxed{T_{trans} = \frac{1}{2} \vec{V}_{cm} \cdot \vec{P}_{cm}}$$

(where cm = center of mass)

Using the expression

$$\vec{\omega} = \hat{\imath} \omega_x + \hat{\jmath} \omega_y + \hat{k} \omega_z$$

and using equ. (4)

$$T = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \frac{1}{2}\omega_x L_x + \frac{1}{2}\omega_y L_y + \frac{1}{2}\omega_z L_z$$

$$= \frac{1}{2}\omega_x^2 I_{xx} + \frac{1}{2}\omega_y^2 I_{yy} + \frac{1}{2}\omega_z^2 I_{zz}$$

$$-\omega_x\omega_y I_{xy} - \omega_y\omega_z I_{yz} - \omega_z\omega_x I_{zx}$$

And we may write this in compact form as

$$\boxed{T = \frac{1}{2}\sum_{\substack{k=1\\ \ell=1}}^{3} \omega_k \omega_\ell I_{k\ell} = \frac{1}{2}\vec{\omega}\cdot\vec{L}}$$

In many practical situations, a rigid body consists of continuous mass with density ρ, which may not be constant. In such cases

$$I_{xx} = \int_V \rho\,(y^2+z^2)\,dx\,dy\,dz \qquad I_{xy} = \int_V \rho\, xy\,dx\,dy\,dz$$

$$I_{yy} = \int_V \rho\,(x^2+z^2)\,dx\,dy\,dz \qquad I_{yz} = \int_V \rho\, yz\,dx\,dy\,dz$$

$$I_{zz} = \int_V \rho\,(x^2+y^2)\,dx\,dy\,dz \qquad I_{zx} = \int_V \rho\, zx\,dx\,dy\,dz$$

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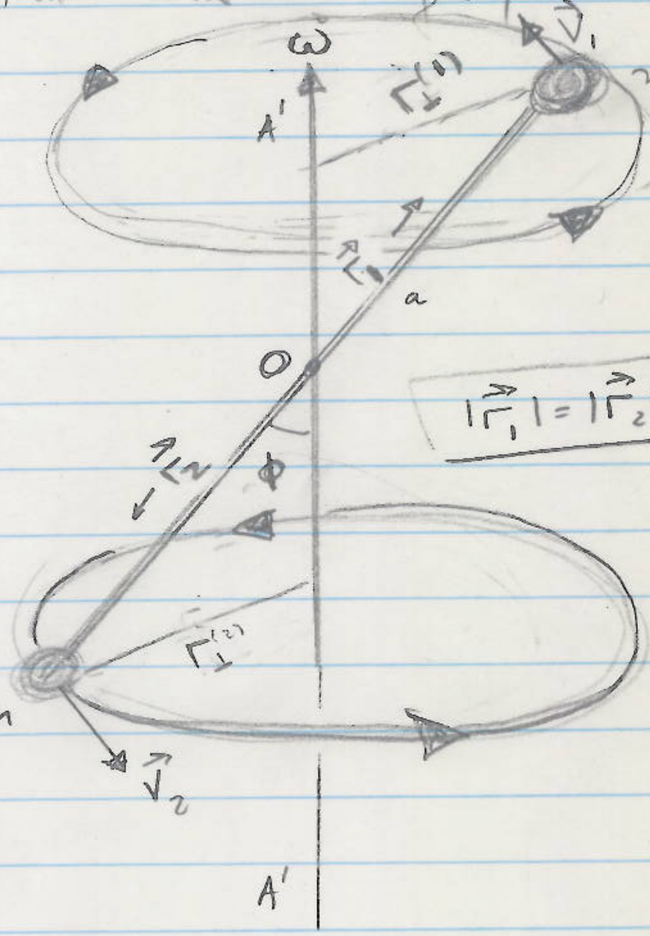
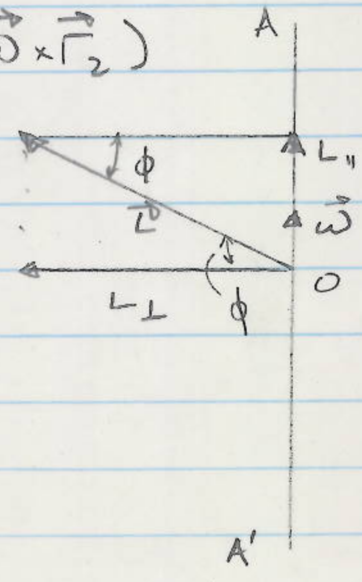
Two point masses of equal mass m are connected by a massless rod of length 2a forming a dumbbell. The dumbbell is constrained to rotate with a constant angular velocity ω about an axis that makes an angle φ with the rod



$|\vec{r}_1| = |\vec{r}_2| = a$

As is shown in the figure, let the dumbbell rotate with an angular velocity ω about an axis AOA' passing through O and lying in an inertial coordinate system. The angular momentum of the system due to the two masses is

$$\vec{L} = \vec{L}_1 + \vec{L}_2$$
$$= m\vec{r}_1 \times (\vec{\omega} \times \vec{r}_1) + m\vec{r}_2 \times (\vec{\omega} \times \vec{r}_2)$$



Note that both $\vec{L}_1$ and $\vec{L}_2$ point in the same direction as $\vec{L}$. It is quite clear that $\vec{L}$ does not point in the same direction a $\vec{\omega}$. That is $L_\perp$ is not zero.

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The magnitude of the angular momentum is

$$\vec{L} = \sum_i \vec{r}_i \times (m_i \vec{v}_i)$$

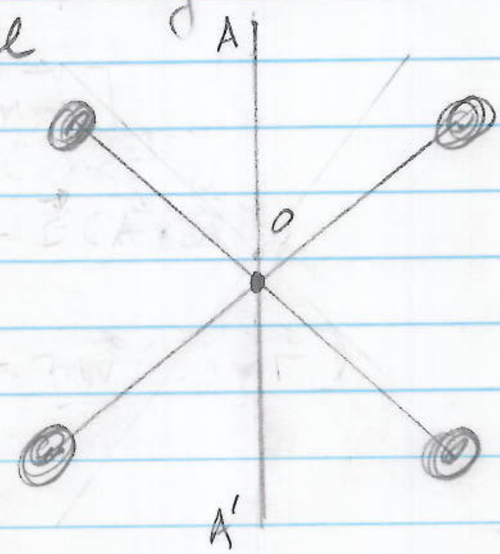
$$|\vec{L}| = ma^2\omega\sin\phi + ma^2\omega\sin\phi = 2ma^2\omega\sin\phi$$
$$= I\omega\sin\phi$$

where I is the moment of inertia of the dumbbell about an axis perpendicular to the length of the connecting rod.

Furthermore, the angular momentum vector $\vec{L}$ is continuously changing direction as it rotates about $\vec{\omega}$. $\vec{L}$ therefore is not constant → it is necessary to apply a torque to maintain this motion

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

If we were to replace the single dumble with a double dumbbell



(That is two dumbbells moving symmetrically)

→ We see $\vec{L}$ and $\vec{\omega}$ will be in the same direction.

We found earlier that

(*)

$$\{I\} = \begin{bmatrix} \sum m_i(x_{i2}^2 + x_{i3}^2) & -\sum m_i x_{i1} x_{i2} & -\sum m_i x_{i1} x_{i3} \\ -\sum m_i x_{i2} x_{i1} & \sum m_i (x_{i1}^2 + x_{i3}^2) & -\sum m_i x_{i2} x_{i3} \\ -\sum m_i x_{i3} x_{i1} & -\sum m_i x_{i3} x_{i2} & \sum m_i (x_{i1}^2 + x_{i2}^2) \end{bmatrix}$$

This is called the moment of inertia tensor or simply

$$\boxed{\text{INERTIA TENSOR}}$$

For a single point particle of mass m

$$\{I\} = m \begin{pmatrix} x_2^2 + x_3^2 & -x_1 x_2 & -x_1 x_3 \\ -x_2 x_1 & x_1^2 + x_3^2 & -x_2 x_3 \\ -x_3 x_1 & -x_3 x_2 & x_1^2 + x_2^2 \end{pmatrix}$$

or in general

$$\{I\} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}$$

Looking at (*) on the previous page we note that we can write the elements of $\{I\}$ as

$$\boxed{I_{kl} = \sum_{i=1}^{n} m_i \left[\delta_{kl} r_i^2 - x_{ik} x_{il} \right]}$$

Now $T_{rot} = \frac{1}{2} \sum_{i=1}^{n} m_i (\vec{\omega} \times \vec{r}_i)^2$

Making use of the vector identity

$$(\vec{A} \times \vec{B})^2 = (\vec{A} \times \vec{B}) \cdot (\vec{A} \times \vec{B}) = A^2 B^2 - (\vec{A} \cdot \vec{B})^2$$

we then have

$$T_{rot} = \frac{1}{2} \sum_{i=1}^{n} m_i \left[\omega^2 r_i^2 - \underbrace{(\vec{\omega} \cdot \vec{r}_i)(\vec{\omega} \cdot \vec{r}_i)}_{(\vec{\omega} \cdot \vec{r}_i)^2} \right]$$

Now $\vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$

$$\vec{\omega} \cdot \vec{\omega} = \sum_{k=1}^{3} \omega_k^2 = \omega^2$$

and $|\vec{r}_i|^2 = \vec{r}_i \cdot \vec{r}_i = \sum_{s=1}^{3} x_{is}^2$

$$T_{rot} = \frac{1}{2} \sum_{i=1}^{n} m_i \left[\left(\sum_{k=1}^{3} \omega_k^2 \right) \left(\sum_{s=1}^{3} x_{is}^2 \right) - \left(\sum_{k=1}^{3} \omega_k x_{ik} \right) \left(\sum_{l=1}^{3} \omega_l x_{il} \right) \right]$$

Making use of the expression

$$\omega_k = \sum_l \delta_{kl}\omega_l \quad \text{or} \quad \delta_{kl} = \begin{cases} 1 & k=l \\ 0 & k\neq l \end{cases}$$

We have then

$$T_{rot} = \frac{1}{2}\sum_{i=1}^{n}\sum_{k,l} m_i\left[\omega_k\omega_l\delta_{kl}\left(\sum_s x_{is}^2\right) - \omega_k\omega_l x_{ik}x_{il}\right]$$

Since all points in a rigid body have the same angular velocity, we may factor these out and we have.

$$T_{rot} = \frac{1}{2}\sum_{k,l}\omega_k\omega_l \underbrace{\sum_i m_i\left[\delta_{kl}\sum_s x_{is}^2 - x_{ik}x_{il}\right]}_{I_{kl}}$$

$$I_{kl} = \sum_{i=1}^{n} m_i(\delta_{kl}r_i^2 - x_{ik}x_{il})$$

Rotational Kinetic Energy becomes:

$$T = \frac{1}{2}\sum_{k,l} I_{kl}\,\omega_k\omega_l$$

<u>Example</u> Let us calculate the components of the moment of inertia tensor for the following configuration:

Point masses of 1, 2, 3, and 4 units are located at (1, 0, 0), (1, 1, 0), (1, 1, 1), and (1, 1, -1)

$$r_i^2 = (x_{i1}^2 + x_{i2}^2 + x_{i3}^2)$$

$$r_1^2 = 1 \qquad r_2^2 = 2 \qquad r_3^2 = 3 \qquad r_4^2 = 3$$

$$\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I_{kj} = \sum_{i=1}^{n} m_i \left[\delta_{kj} (r_i)^2 - x_{ik} x_{ij} \right]$$

$$I_{11} = m_1 (r_1^2 - x_{11}^2) + m_2 (r_2^2 - x_{21}^2) + m_3 (r_3^2 - x_{31}^2) + m_4 (r_4^2 - x_{41}^2)$$

$$= 1(0) + 2(2-1) + 3(3-1) + 4(3-1)$$

$$= 2 + 6 + 8 = \underline{16}$$

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$$I_{12} = I_{21} = m_1(-x_{11}x_{12}) + m_2(-x_{21}x_{22}) + m_3(-x_{31}x_{32}) + m_4(-x_{41}x_{42})$$

$$= (1)[-(1)(0)] + 2[-(1)(1)] + 3[-(1)(1)] + 4[-(1)(1)]$$
$$= -2-3-4 = \underline{-9}$$

$$I_{22} = m_1(r_1^2 - x_{12}^2) + m_2(r_2^2 - x_{22}^2) + m_3(r_3^2 - x_{32}^2) + m_4(r_4^2 - x_{42}^2)$$

$$= 1(1-0) + 2(2-1) + 3(3-1) + 4(3-1)$$
$$= 1 + 2 + 6 + 8 = \underline{17}$$

$$I_{23} = I_{32} = m_1(-x_{12}x_{13}) + m_2(-x_{22}x_{23}) + m_4(-x_{32}x_{33}) + m_4(-x_{42}x_{43})$$

$$= 1[-(0)(0)] + 2[-(1)(0)] + 3[-(1)(1)] + 4[-(1)(-1)]$$
$$= -3 + 4 = \underline{1}$$

$$I_{13} = I_{31} = m_1(-x_{11}x_{13}) + m_2(-x_{21}x_{23}) + m_3(-x_{31}x_{33}) + m_4(-x_{41}x_{43})$$

$$= 1[-(1)(0)] + 2[-(1)(0)] + 3[-(1)(1)] + 4[-(1)(-1)]$$
$$= -3 + 4 = \underline{1}$$

$$I_{33} = m_1(r_1^2 - x_{13}^2) + m_2(r_2^2 - x_{23}^2) + m_3(r_3^2 - x_{33}^2) + m_4(r_4^2 - x_{43}^2)$$
$$= 1(1-0) + 2(2-0) + 3(3-1) + 4(3-1)$$
$$= 1 + 4 + 6 + 8 = 19$$

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$$\{I\} = \begin{bmatrix} 16 & -9 & 1 \\ -9 & 17 & 1 \\ 1 & 1 & 19 \end{bmatrix}$$

We shall seek to determine the principal axes

Determination of Principal Axes

We are given the moment of inertia and the product of inertia elements of a rigid body in terms of an arbitrarily chosen coordinate axes through point O. We wish to find the principal axes about the origin at O. We call this process:

DIAGONALIZING THE MATRIX TENSOR

We make use of the fact that if the rotation axis is the principal axis then both the angular momentum $\vec{L}$ and the angular velocity $\vec{\omega}$ are directed along this AXES and HENCE ARE PROPORTIONAL TO ONE ANOTHER. If I is the moment of inertia about the axes, we may write

$$\vec{L} = I\vec{\omega} = I\omega_x\hat{i} + I\omega_y\hat{j} + I\omega_z\hat{k}$$

We compare with equ (4) from p. 11-3

$$L_x = I\omega_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$L_y = I\omega_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$L_z = I\omega_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

After rearranging

$$\begin{array}{cccc}
(I_{xx} - I)\,\omega_x & I_{xy}\,\omega_y & I_{xz} & = 0 \\
I_{yx}\,\omega_x & (I_{yy} - I)\,\omega_y & I_{yz}\,\omega_z & = 0 \\
I_{zx}\,\omega_x & I_{zy}\,\omega_y & (I_{zz} - I)\,\omega_z & = 0
\end{array}$$

For these equations to have nontrivial solutions, the determinant of the coefficients must vanish

$$\begin{vmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I \end{vmatrix} = 0$$

This is a secular or characteristic equation and is cubic in the form of

$$-I^3 + A I^2 + B I + C = 0$$

Here A, B, and C are constants and depend on the values of the moment of inertia and product of inertia elements. Each of the three roots,

$$I_x, I_y, \text{ and } I_z \text{ (or } I_1, I_2, \text{ and } I_3\text{)},$$

corresponds to the moment of inertia about one of the principal axes. These values of I_x, I_y and I_z are called the

$$\boxed{\begin{array}{c}\text{PRINCIPAL MOMENTS}\\ \text{of}\\ \text{INERTIA}\end{array}}$$

Let us first do an example. We shall develop this topic further after the example.

$$\{I\} = \begin{bmatrix} 13 & -2 & 1 \\ -2 & 16 & 4 \\ 1 & 4 & 15 \end{bmatrix}$$

Let us find the principal axes.

$$\begin{vmatrix} 13-I & -2 & 1 \\ -2 & 16-I & 4 \\ 1 & 4 & 15-I \end{vmatrix} = 0$$

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we have:

$$\left|\begin{array}{ccc:cc} 13-I & -2 & 1 & 13-I & -2 \\ -2 & 16-I & 4 & -2 & 16-I \\ 1 & 4 & 15-I & 1 & 4 \end{array}\right|$$

$$(13-I)\left[(16-I)(15-I)-16\right]+(-2)\left[(4)(1)-(-2)(15-I)\right]$$
$$(1)\left[(-2)(4)-(16-I)(1)\right]$$

$$(13-I)\left[240-31I+I^2-16\right]+(-2)\left[4+30-2I\right]$$
$$(1)\left[-8+I-16\right]$$

$$(13-I)\left[I^2-31I+224\right]+(-2)\left[34-2I\right]$$
$$+\left[I-24\right]$$

$$13I^2-403I+2912-I^3+31I^2-224I-68+4I$$
$$+I-24$$

$$-I^3+44I^2-622I+2820=0$$

$$\Rightarrow\quad I^3-44I^2+622I-2820=0$$

$$I^3+aI^2+bI+c=0$$

$a=-44$ $\qquad$ $b=+622$ $\qquad$ $c=-2820$

5.6 Quadratic and Cubic Equations

The roots of simple algebraic equations can be viewed as being functions of the equations' coefficients. We are taught these functions in elementary algebra. Yet, surprisingly many people don't know the right way to solve a quadratic equation with two real roots, or to obtain the roots of a cubic equation.

There are two ways to write the solution of the *quadratic equation*

$$ax^2 + bx + c = 0 \tag{5.6.1}$$

with real coefficients $a, b, c,$ namely

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{5.6.2}$$

and

$$x = \frac{2c}{-b \pm \sqrt{b^2 - 4ac}} \tag{5.6.3}$$

If you use *either* (5.6.2) *or* (5.6.3) to get the two roots, you are asking for trouble: If either a or c (or both) are small, then one of the roots will involve the subtraction of b from a very nearly equal quantity (the discriminant); you will get that root very inaccurately. The correct way to compute the roots is

$$q \equiv -\frac{1}{2}\left[b + \operatorname{sgn}(b)\sqrt{b^2 - 4ac}\right] \tag{5.6.4}$$

Then the two roots are

$$x_1 = \frac{q}{a} \qquad \text{and} \qquad x_2 = \frac{c}{q} \tag{5.6.5}$$

If the coefficients $a, b, c,$ are complex rather than real, then the above formulas still hold, except that in equation (5.6.4) the sign of the square root should be chosen so as to make

$$\operatorname{Re}(b^* \sqrt{b^2 - 4ac}) \geq 0 \tag{5.6.6}$$

where Re denotes the real part and asterisk denotes complex conjugation.

Apropos of quadratic equations, this seems a convenient place to recall that the inverse hyperbolic functions $\sinh^{-1}$ and $\cosh^{-1}$ are in fact just logarithms of solutions to such equations,

$$\sinh^{-1}(x) = \quad \ln\left(x + \sqrt{x^2 + 1}\right) \tag{5.6.7}$$

$$\cosh^{-1}(x) = \pm \ln\left(x + \sqrt{x^2 - 1}\right) \tag{5.6.8}$$

Equation (5.6.7) is numerically robust for $x \geq 0$. For negative x, use the symmetry $\sinh^{-1}(-x) = -\sinh^{-1}(x)$. Equation (5.6.8) is of course valid only for $x \geq 1$. Since FORTRAN mysteriously omits the inverse hyperbolic functions from its list of intrinsic functions, equations (5.6.7)–(5.6.8) are sometimes quite essential.

For the *cubic equation*

$$x^3 + ax^2 + bx + c = 0 \tag{5.6.9}$$

with real or complex coefficients a, b, c, first compute

$$Q \equiv \frac{a^2 - 3b}{9} \qquad \text{and} \qquad R \equiv \frac{2a^3 - 9ab + 27c}{54} \tag{5.6.10}$$

If Q and R are real (always true when a, b, c are real) *and* $R^2 < Q^3$, then the cubic equation has three real roots. Find them by computing

$$\theta = \arccos(R/\sqrt{Q^3}) \tag{5.6.11}$$

in terms of which the three roots are

$$\begin{aligned} x_1 &= -2\sqrt{Q}\cos\left(\frac{\theta}{3}\right) - \frac{a}{3} \\ x_2 &= -2\sqrt{Q}\cos\left(\frac{\theta + 2\pi}{3}\right) - \frac{a}{3} \\ x_3 &= -2\sqrt{Q}\cos\left(\frac{\theta - 2\pi}{3}\right) - \frac{a}{3} \end{aligned} \tag{5.6.12}$$

(This equation first appears in Chapter VI of François Viète's treatise "De emendatione," published in 1615!)

Otherwise, compute

$$A = -\left[R + \sqrt{R^2 - Q^3}\right]^{1/3} \tag{5.6.13}$$

where the sign of the square root is chosen to make

$$\mathrm{Re}(R^* \sqrt{R^2 - Q^3}) \geq 0 \tag{5.6.14}$$

(asterisk again denoting complex conjugation). If Q and R are both real, equations (5.6.13)–(5.6.14) are equivalent to

$$A = -\mathrm{sgn}(R)\left[|R| + \sqrt{R^2 - Q^3}\right]^{1/3} \tag{5.6.15}$$

where the positive square root is assumed. Next compute

$$B = \begin{cases} Q/A & (A \neq 0) \\ 0 & (A = 0) \end{cases} \tag{5.6.16}$$

in terms of which the three roots are

$$x_1 = (A + B) - \frac{a}{3} \tag{5.6.17}$$

(the single real root when a, b, c are real) and

$$x_2 = -\frac{1}{2}(A+B) - \frac{a}{3} + i\frac{\sqrt{3}}{2}(A-B)$$
$$x_3 = -\frac{1}{2}(A+B) - \frac{a}{3} - i\frac{\sqrt{3}}{2}(A-B) \tag{5.6.18}$$

(in that same case, a complex conjugate pair). Equations (5.6.13)–(5.6.16) are arranged both to minimize roundoff error, and also (as pointed out by A.J. Glassman) to ensure that no choice of branch for the complex cube root can result in the spurious loss of a distinct root.

If you need to solve many cubic equations with only slightly different coefficients, it is more efficient to use Newton's method (§9.4).

CITED REFERENCES AND FURTHER READING:

Weast, R.C. (ed.) 1967, *Handbook of Tables for Mathematics*, 3rd ed. (Cleveland: The Chemical Rubber Co.), pp. 130–133.

Pachner, J. 1983, *Handbook of Numerical Analysis Applications* (New York: McGraw-Hill), §6.1.

McKelvey, J.P. 1984, *American Journal of Physics*, vol. 52, pp. 269–270; see also vol. 53, p. 775, and vol. 55, pp. 374–375.

5.7 Numerical Derivatives

Imagine that you have a procedure which computes a function $f(x)$, and now you want to compute its derivative $f'(x)$. Easy, right? The definition of the derivative, the limit as $h \to 0$ of

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \tag{5.7.1}$$

practically suggests the program: Pick a small value h; evaluate $f(x+h)$; you probably have $f(x)$ already evaluated, but if not, do it too; finally apply equation (5.7.1). What more needs to be said?

Quite a lot, actually. Applied uncritically, the above procedure is almost guaranteed to produce inaccurate results. Applied properly, it can be the right way to compute a derivative only when the function f is *fiercely* expensive to compute, when you already have invested in computing $f(x)$, and when, therefore, you want to get the derivative in no more than a single additional function evaluation. In such a situation, the remaining issue is to choose h properly, an issue we now discuss:

There are two sources of error in equation (5.7.1), truncation error and roundoff error. The truncation error comes from higher terms in the Taylor series expansion,

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \frac{1}{6}h^3 f'''(x) + \cdots \tag{5.7.2}$$

whence

$$\frac{f(x+h) - f(x)}{h} = f' + \frac{1}{2}hf'' + \cdots \tag{5.7.3}$$

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p. 179 of Numerical Recipes (cubic equations)

$$Q = \frac{a^2 - 3b}{9} = \frac{1}{9}\left[(-44)^2 - (3)(622)\right]$$

$\underline{Q = 7.78}$

$$R = \frac{1}{54}\left[2a^3 - 9ab + 27c\right]$$

$$= \frac{1}{54}\left[2(-44)^3 - 9(-44)(622) + 27(-2820)\right]$$

$$= \frac{1}{54}\left[-170{,}368 + 246\,312 - 76\,140\right]$$

$\underline{R = -3.63}$

Is $R^2 < Q^3$ $\underline{\text{Yes}}$ so the cubic equation has three roots

$[Q^3 = 470.91]$

$$\theta = \cos^{-1}\left[\frac{-3.63}{[470.91]^{1/2}}\right] = 99.63°$$

$$I_1 = -2\sqrt{Q}\cos\frac{\theta}{3} - \frac{a}{3} = -2[7.78]^{1/2}\cos(33.21°) - \frac{(-44)}{3}$$

$\boxed{I_1 = +10.0}$

$$I_2 = -2\sqrt{Q}\cos\left(\frac{\theta + 2\pi}{3}\right) - \frac{a}{3}$$

$$\boxed{I_2 = 19.65}$$

$$I_3 = -2\sqrt{Q}\cos\left(\frac{\theta - 2\pi}{3}\right) - \frac{a}{3}$$

$$\boxed{I_3 = 14.35}$$

The direction of any one principal axis is determined by substituting for I any one of the three roots I_1, I_2, or I_3, in $\vec{L} = I\vec{\omega}$. Then both $\vec{L}$ and $\vec{\omega}$ are directed along this axis.

The direction of $\vec{\omega}$ wrt the body coordinate system is then the same as the direction of the principal axes for this body.

i.e $\vec{L} = I_1\vec{\omega} \Rightarrow \vec{\omega}_1 \parallel \vec{L}_1$.

The elements of the principal moment of inertia are called the <u>EIGENVALUES</u>. The directions of the principal axes are called the <u>EIGENVECTORS</u>.

Let us now find the principal axes

$$\begin{bmatrix} 13-I_i & -2 & 1 \\ -2 & 16-I_i & 4 \\ 1 & 4 & 15-I_i \end{bmatrix} \begin{bmatrix} \omega_{1i} \\ \omega_{2i} \\ \omega_{3i} \end{bmatrix} = 0$$

$I_i = (10.0, 19.65, 14.35)$

for $i = 1$

$$\begin{bmatrix} 3 & -2 & 1 \\ -2 & 6 & 4 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} \omega_{11} \\ \omega_{21} \\ \omega_{31} \end{bmatrix} = 0$$

$$\Rightarrow \quad 3\omega_{11} - 2\omega_{21} + \omega_{31} = 0 \quad (1)$$
$$-2\omega_{11} + 6\omega_{21} + 4\omega_{31} = 0 \quad (2)$$
$$\omega_{11} + 4\omega_{21} + 5\omega_{31} = 0 \quad (3)$$

Multiply equation (1) by -4 and adding to (2)

$$-14\omega_{11} + 14\omega_{21} = 0 \quad \Rightarrow \quad \boxed{\omega_{11} = \omega_{21}}$$

Now substituting this into equation (3) gives

$$5\omega_{11} + 5\omega_{31} = 0 \quad \Rightarrow \quad \boxed{\omega_{31} = -\omega_{11}}$$

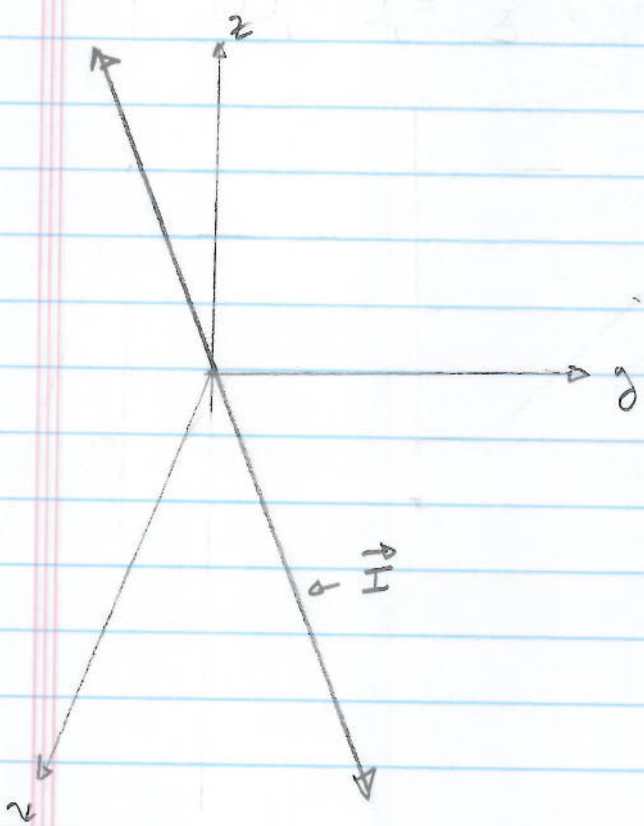
This means $\omega_{11} : \omega_{21} : \omega_{31} = 1 : 1 : -1$

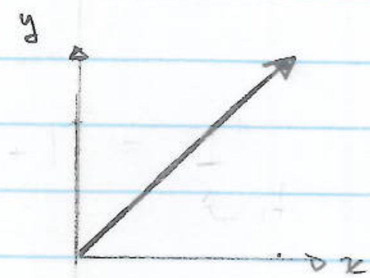
11-23

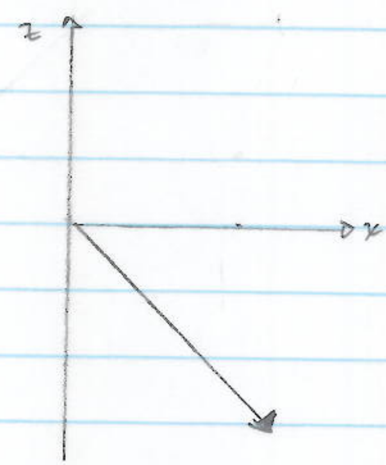
This means the principal axis associated with I_1 is

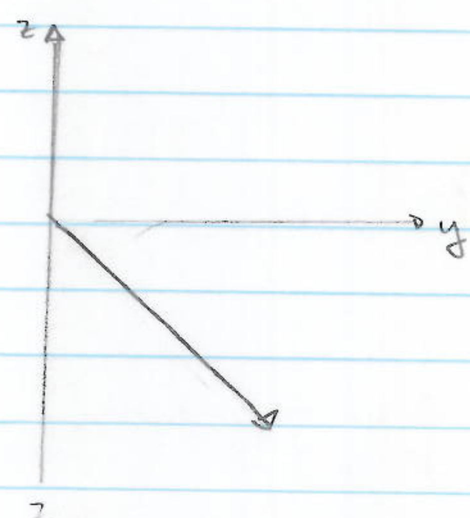
$$\frac{1}{\sqrt{3}}(\hat{x}+\hat{y}-\hat{z}) = \frac{\vec{I}_1}{|\vec{I}_1|} = \hat{I}_1$$

Normalization is $(\omega_{11}^2+\omega_{21}^2+\omega_{31}^2)^{-1/2}$







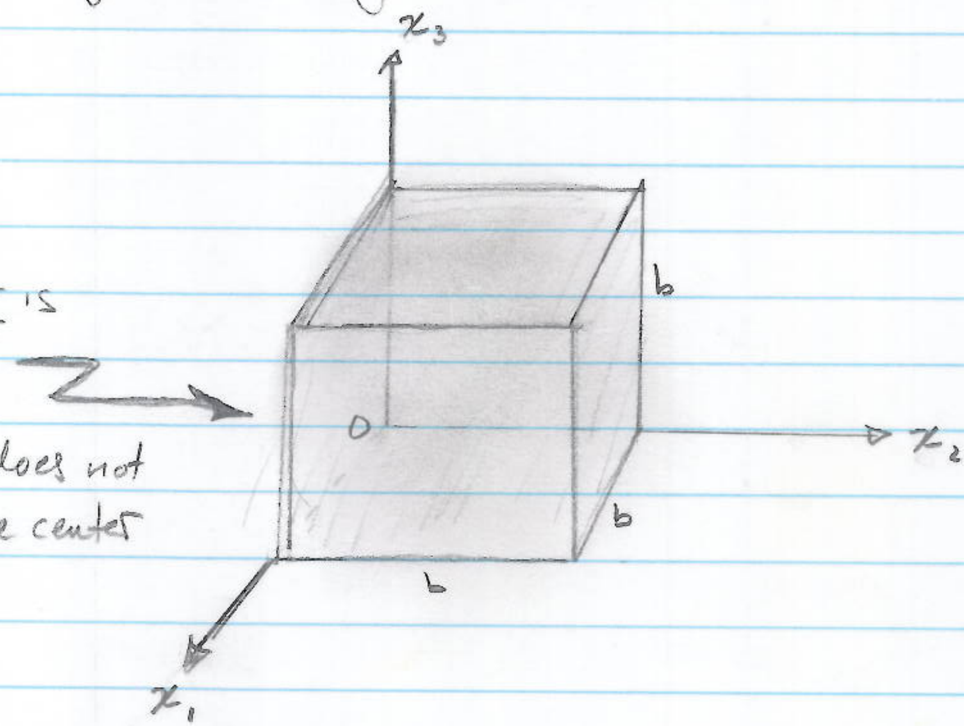


11-24

Let us calculate the inertia tensor of a homogeneous cube of density ρ, mass M, and side of length b. Let one corner be the origin and let three adjacent edges lie along the coordinate axes.

PIVOT POINT IS At the corner

The origin does not lie at the center of mass.



$$I_{ij} = \int_V \rho(\vec{r}) \left(\delta_{ij} \sum_k x_k^2 - x_i x_j \right) dx_1 dx_2 d_3$$

$\rho = \text{const}$ ✓

$\Rightarrow$

$$I_{11} = \rho \int_0^b dx_3 \int_0^b dx_2 \left(x_2^2 + x_3^2 \right) \int_0^b dx_1$$

$$= \frac{2}{3} \rho b^5 \quad \text{but} \quad M = \rho V = \rho b^3$$

$$\boxed{I_{11} = \frac{2}{3} M b^2}$$

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By symmetry all diagonal elements are equal (likewise all off-diagonal elements are equal

$$I_{12} = -\rho \int_0^b dx_2 \int_0^b dx_1\, x_1 x_2 \int_0^b dx_3$$

$$= -\frac{1}{4}\rho b^5 = -\frac{1}{4} M b^2$$

$$\{I\} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{bmatrix} M b^2$$

Now let us calculate the principal moments of inertia:

$$\begin{vmatrix} 2/3 - I & -1/4 & -1/4 \\ -1/4 & 2/3 - I & -1/4 \\ -1/4 & -1/4 & 2/3 - I \end{vmatrix} = 0$$

$$\boxed{I^3 - 2I^2 + \frac{55}{48} I - \frac{121}{864} = 0}$$

And solving for I gives the roots:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 11/12 \\ 11/12 \end{bmatrix}$$

← You can check this by substituting into the highlighted equation.

Now let us find the principal axes:

$\underline{I_1 = 1/6}$:

$$\begin{bmatrix} (2/3 - 1/6) & -1/4 & -1/4 \\ -1/4 & (2/3 - 1/6) & -1/4 \\ -1/4 & -1/4 & (2/3 - 1/6) \end{bmatrix} \begin{bmatrix} \omega_{11} \\ \omega_{21} \\ \omega_{31} \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} \omega_{11} \\ \omega_{21} \\ \omega_{31} \end{bmatrix} = 0$$

$$\Rightarrow \quad \begin{aligned} 2\omega_{11} - \omega_{21} - \omega_{31} &= 0 \\ -\omega_{11} + 2\omega_{21} - \omega_{31} &= 0 \\ -\omega_{11} - \omega_{21} + 2\omega_{31} &= 0 \end{aligned}$$

$$\Rightarrow \quad \omega_{11} = \omega_{21} = \omega_{31} \Rightarrow 1:1:1$$

$$\hat{I}_1 = \frac{1}{\sqrt{3}}(\hat{x} + \hat{y} + \hat{z})$$ ← A DIAGONAL!