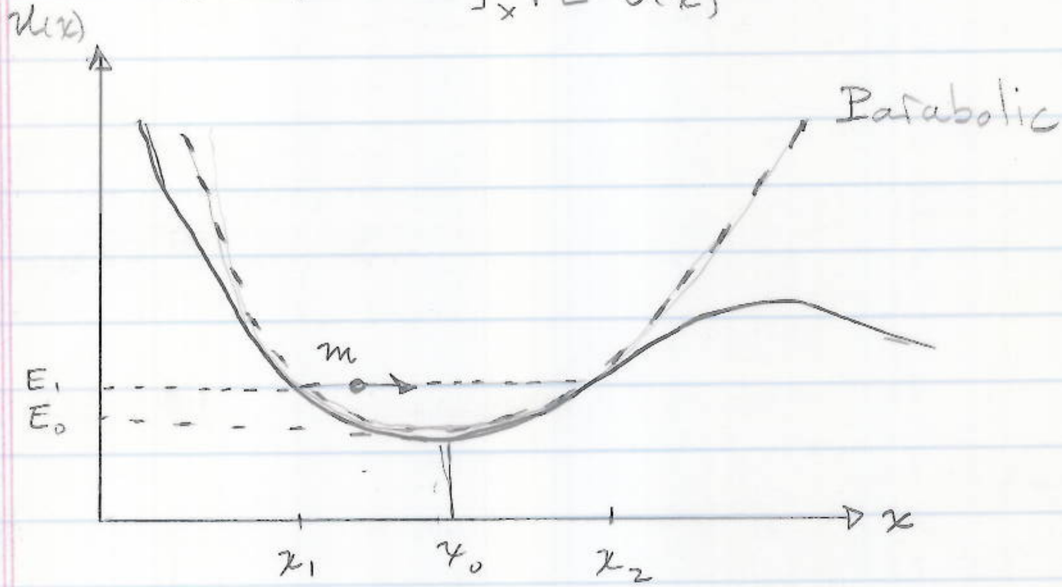
Oscillations

We derived:

$$t_2 - t_1 = \sqrt{\frac{m}{2}} \int_{x}^{x_2} \frac{dx}{\sqrt{E - U(x)}}$$



A particle of mass m and energy E is moving in an arbitrary potential energy function $U(x)$ shown by the solid heavy curve. The dotted curve is the parabolic potential approximation of an arbitrary potential.

The Period of oscillation is

$$T = 2(t_2 - t_1) = \sqrt{2m} \int_{x_1}^{x_2} \frac{dx}{\sqrt{E - U(x)}}$$

Suppose a particle is oscillating about a point of stable equilibrium x_0. Let us expand the potential function $U(x)$ in a Taylor series about the point x_0

$$U(x) = U(x_0) + \left(\frac{dU}{dx}\right)_{x=x_0}(x-x_0) + \frac{1}{2}\left(\frac{d^2U}{dx^2}\right)_{x=x_0}(x-x_0)^2$$
$$+ \frac{1}{6}\left(\frac{d^3U}{dx^3}\right)_{x=x_0}(x-x_0)^3 + \frac{1}{24}\left(\frac{d^4U}{dx^4}\right)_{x=x_0}(x-x_0)^4 + \cdots$$

Let's only consider small displacements in symmetrical potentials. The term $U(x_0)$ is a constant term and can be dropped without affecting the results. Also, since x_0 is a point of minimum, for stable equilibrium in a symmetric potential, the odd terms must be zero.
[Note that if the expression resulting from the expansion of $F(x)$ were used, the even terms would be zero – Why?]

Therefore

equilibrium point

$$\left(\frac{dU}{dx}\right)_{x=x_0} = 0 \qquad \left(\frac{d^3U}{dx^3}\right)_{x=x_0} = 0$$

while

$$\left(\frac{d^2U}{dx^2}\right)_{x=x_0} > 0$$

Let us define $(x-x_0) = x'$

$$\left(\frac{d^2U}{dx^2}\right)_{x=x_0} = k$$

$$\frac{1}{6}\left(\frac{d^4U}{dx^4}\right)_{x=x_0} = \epsilon$$

The potential may then be written

$$U(x') = \frac{1}{2}kx'^2 + \frac{1}{4}\epsilon x'^4 + \ldots$$

Let us assume that the origin is located at the equilibrium point so that $x_0 = 0$ and $x' = x$, and by neglecting higher order terms, we get

$$U(x) = \frac{1}{2}kx^2 + \frac{1}{4}\epsilon x^4$$

Furthermore, since the motion of the particle is in a conservative field

$$F(x) = -\frac{dU}{dx}$$

$$\Rightarrow F(x) = -kx - \epsilon x^3$$

SIMPLE HARMONIC OSCILLATOR

$$m\ddot{x} = -kx$$

$$\boxed{\ddot{x} + \omega_0^2 x = 0} \qquad \omega_0^2 = k/m$$

free natural angular frequency

The solution of this equation is:

$$x(t) = A\sin(\omega_0 t - \delta)$$

$$x(t) = A\cos(\omega_0 t - \phi)$$

Here the phase δ & ϕ differ by $\pi/2$

Energy of a SHO.

$$x = A\sin(\omega_0 t - \delta)$$

$$\dot{x} = \omega_0 A\cos(\omega_0 t - \delta)$$

The maximum value of the velocity is

$$V_0 = \omega_0 A = \sqrt{\frac{k}{m}}\,A$$

Note: I use K instead of T as I do not wish to confuse Kinetic Energy & Period

Hence the kinetic energy K of the oscillator is

$$K = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\omega_0^2 A^2\cos^2(\omega_0 t - \delta)$$
$$= K_0\cos^2(\omega_0 t - \delta)$$

where K_0 is the maximum kinetic energy given by

$$K_0 = \frac{1}{2}m\omega_0^2 A^2 = \frac{1}{2}kA^2$$

The potential energy of the system is equal to the work done by the applied force $F_a = -F = -(-kx) = kx$ in displacing the system from $x = 0$ to $x = x$

$$U(x) = W = \int_0^x F_a\,dx = \int_0^x kx\,dx = \frac{1}{2}kx^2$$

dW is the incremental work necessary to move the particle by an amount dx against the restoring force F

3-5

Because $x = A\sin(\omega_0 t - \delta)$

$$U(x) = \frac{1}{2}kA^2\sin^2(\omega_0 t - \delta) = U_0\sin^2(\omega_0 t - \delta)$$

where U_0 is the maximum potential energy

Thus, the total mechanical energy, which is a constant for a conservative force, is

$$E = K + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

$$\dot{x} = \pm\left(\frac{2E}{m} - \frac{k}{m}x^2\right)^{1/2}$$

$$\pm\int\frac{dx}{\sqrt{\left(\frac{2E}{k}\right) - x^2}} = \sqrt{\frac{k}{m}}\int dt$$

And the solutions are:

$$x = A\sin(\omega_0 t + \phi_1)$$
$$x = A\cos(\omega_0 t + \phi_2)$$

where ϕ_1 and ϕ_2 are constants, while the amplitude is given by

$$A = \sqrt{\frac{2E}{k}}$$

This relation tells us that x can vary between $+A$ & $-A$, that is $-\sqrt{\frac{2E}{k}} \leq x \leq +\sqrt{\frac{2E}{k}}$

3-6

To find the average values of U and K over one complete time period, we use the following general expression for the average value of quantity $f(t)$

$$\langle f \rangle = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} f(t)\, dt$$

That is

$$\langle U \rangle = \frac{\int_0^T U\, dt}{\int_0^T dt} = \frac{\int_0^T U_0 \sin^2(\omega_0 t + \phi)\, dt}{T}$$

$$= \frac{1}{2} U_0 = \frac{1}{4} k A^2$$

and similarly $\quad \langle K \rangle = \frac{1}{2} K_0 = \frac{1}{4} k A^2$

That is $\quad \langle U \rangle = \langle K \rangle = \frac{1}{2} E$ ✓

If, instead of time averages, we calculate space averages over one complete time period, we get

$$\langle U \rangle_{space} = \frac{1}{6} k A^2, \qquad \langle K \rangle_{space} = \frac{1}{3} k A^2$$

and

$$\langle E \rangle_{space} = \langle V \rangle_{space} + \langle K \rangle_{space} = \langle E \rangle_{Time}$$

(c.f. problem 3-4)

3-7

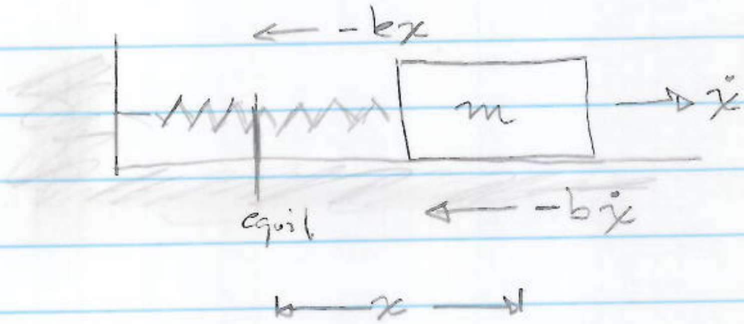
Damped Harmonic Oscillator

In practice, in any physical system there will be dissipative or damping forces; the oscillating system will lose energy with time and come to rest.

Let us imagine a mass moving in a viscous medium such as in a fluid or air. As long as the speed of the mass is small (so as not to cause turbulence), the damping force will be proportional to its velocity

$$F_d = -bv = -b\dot{x}$$

b is a positive constant



$$F_{net} = F + F_d = -kx - b\dot{x}$$

From Newton's second, we have:

$$m\ddot{x} + b\dot{x} + kx = 0$$

3-8

Let us rewrite this second order differential equation as:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

where $\gamma = \frac{b}{2m}$ and $\omega_0^2 = \frac{k}{m}$

Let us try the trial solution

$$x = e^{\lambda t}$$

$$\dot{x} = \lambda e^{\lambda t}$$

$$\ddot{x} = \lambda^2 e^{\lambda t}$$

and substituting this trial solution into the above equation we obtain

$$e^{\lambda t}(\lambda^2 + 2\gamma\lambda + \omega_0^2) = 0$$

Since $e^{\lambda t} \neq 0$, we must have

$$\lambda^2 + 2\gamma\lambda + \omega_0^2 = 0$$

The auxiliary equation has the roots

$$\lambda_1 = -\gamma + \sqrt{\gamma^2 - \omega_0^2}$$

$$\lambda_2 = -\gamma - \sqrt{\gamma^2 - \omega_0^2}$$

Thus the general solution of this second order differential equation is

$$x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

or

$$\Longrightarrow \quad x(t) = e^{-\gamma t}\left(A_1 e^{+\sqrt{\gamma^2-\omega_0^2}\, t} + A_2 e^{-\sqrt{\gamma^2-\omega_0^2}\, t}\right)$$

The following three cases of this solution are of especial interest and will be discussed in some detail.

- Case I — UNDERDAMPED (oscillatory) — $\omega_0^2 > \gamma^2$ — λ_1 and λ_2 are imaginary roots
- Case II — Critically damped (not oscillatory) — $\omega_0^2 = \gamma^2$ — λ_1 and λ_2 are real and equal roots
- Case III — Overdamped (not oscillatory) — $\omega_0^2 < \gamma^2$ — λ_1 and λ_2 are real roots

CASE I — Underdamped Oscillations

$$\omega_0^2 > \gamma^2$$

We shall make the convenient substitution:

$$\omega_1 = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

3-10

The arguments of the exponential in $\Longrightarrow$ are then imaginary, and we may then write the equation as.

$$x(t) = e^{-\gamma t}(A_1 e^{+i\omega_1 t} + A_2 e^{-i\omega_1 t})$$

which is the solution of an underdamped oscillator
Using the relation

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

we may rewrite the above as:

$$x(t) = e^{-\gamma t}\left[(A_1+A_2)\cos\omega_1 t + i(A_1-A_2)\sin\omega_1 t\right]$$

Making the substitutions:

$$B = i(A_1-A_2) \quad \& \quad C = (A_1+A_2)$$

we may write

$$x(t) = e^{-\gamma t}\left[B\sin\omega_1 t + C\cos\omega_1 t\right]$$

This may still be written in a slightly different form

From Exercise D-6 in Appendix D

And making the following substitutions.

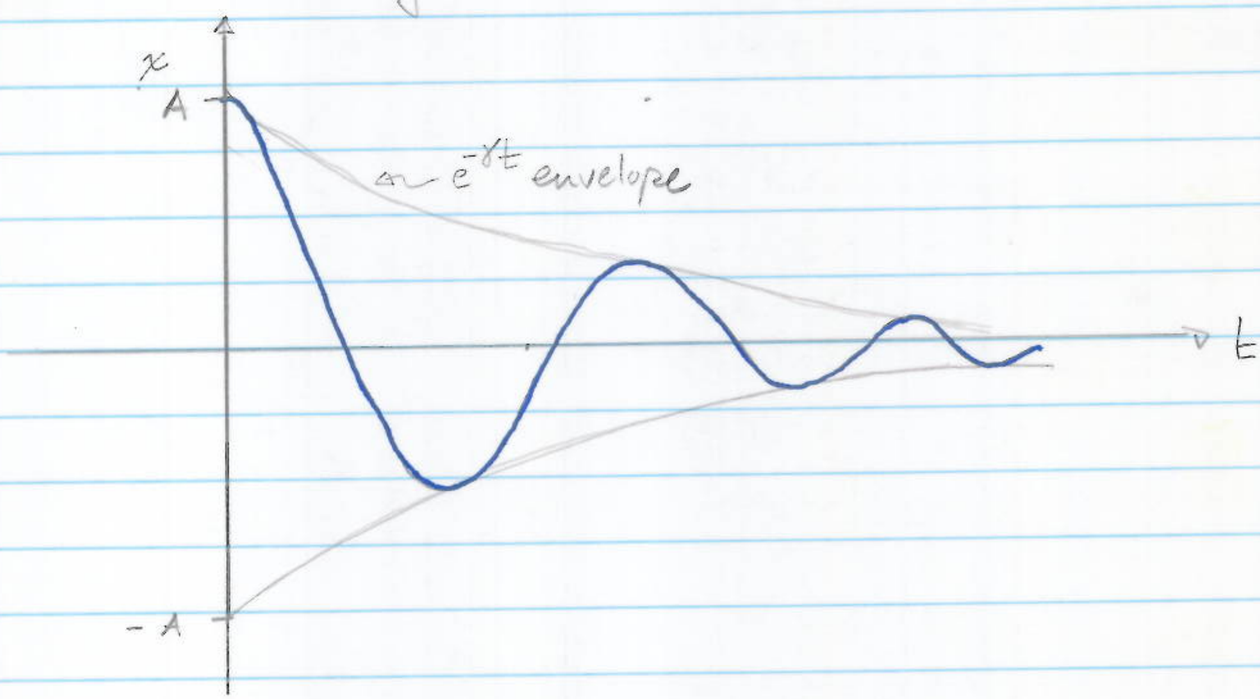
$$A = \sqrt{B^2 + C^2} \quad \text{and} \quad \tan\phi = -\frac{C}{B}$$

we obtain

$$\Rightarrow \quad x(t) = A e^{-\gamma t} \cos(\omega_1 t + \phi) \qquad (*)$$

A and ϕ are real quantities.

The solution given by the equation above indicates that for the damped oscillator, the motion is oscillatory, but the amplitude decays exponentially.



The maximum amplitude of the oscillations decreases exponentially with time due to the factor $e^{-\gamma t}$, and lies between the two curves $A(t) = \pm A e^{-\gamma t}$

3-13

Let us further investigate the solution to the damped harmonic oscillator

$$x(t) = A e^{-\gamma t} \cos(\omega_1 t + \phi)$$

Here, (again) $\gamma = \frac{b}{2m}$ and $\omega_1 = \sqrt{\omega_0^2 - \gamma^2}$

If γ is small, then $\omega_1 \simeq \omega_0$ and we may call ω_1 the "angular frequency".

For $\gamma^2 \ll \omega_0^2$, then by means of the binomial expansion:

$$\omega_1 = \omega_0 \sqrt{1 - \frac{\gamma^2}{\omega_0^2}} \approx \omega_0 \left[1 - \frac{1}{2} \frac{\gamma^2}{\omega_0^2} + \cdots \right.$$

$$\simeq \omega_0 - \frac{\gamma^2}{2\omega_0}$$

For $\gamma \ll \omega_0 \Rightarrow \omega_1 \simeq \omega_0$

If $\frac{\gamma}{\omega_1} \ll 1$, the amplitude envelope $A_e(t) = A e^{-\gamma t}$ changes very slowly with time – this is a lightly damped system – while the cosine term in $x(t)$ makes several zero crossings.

On the other hand for $\frac{\gamma}{\omega_1} \gg 1$, the system is said to be heavily damped because $A_e(t)$ will decrease very rapidly and goes to zero, while the cosine term makes only a few crossings in this time.

In either case of lightly or heavily damped systems

$$\frac{Ae^{-\gamma t_1}}{Ae^{-\gamma t_2}} = \frac{Ae^{-\gamma t_m}}{Ae^{-\gamma(t_m+T_1)}} = e^{\gamma T_1}$$

where $t_1 = t_m$ is the time where the first maximum occurs and $t_2 = t_m + T_1$ is the time when the next maximum occurs; T_1 being the time period of the damped oscillation.

The quantity $e^{\gamma T_1}$ is called the <u>DECREMENT</u> of motion. Its logarithm, γT_1, is called the <u>logarithmic</u> <u>decrement</u>

$$\delta = \ln e^{\gamma T_1} = \gamma T_1 = \left(\frac{b}{2m}\right)\left(\frac{2\pi}{\omega_1}\right)$$

$$= \frac{b}{m}\frac{\pi}{\omega_1}$$

<u>CASE II</u> Critically Damped Oscillations

$$\omega_0^2 = \gamma^2$$

From our solution $x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$ with the auxiliary equation $\lambda^2 + 2\gamma\lambda + \omega_0^2 = 0$ we found

$$\lambda_1 = -\gamma + \sqrt{\gamma^2 - \omega_0^2}$$

$$\lambda_2 = -\gamma - \sqrt{\gamma^2 - \omega_0^2}$$

For this case, the two roots λ_1 and λ_2 are equal $\lambda_1 = \lambda_2 = -\gamma$ and the prescribed solution takes the form

$$x(t) = (A_1 + A_2)e^{-\gamma t} = \underbrace{B_2}_{(A_1+A_2) = \text{constant.}} e^{-\gamma t}$$

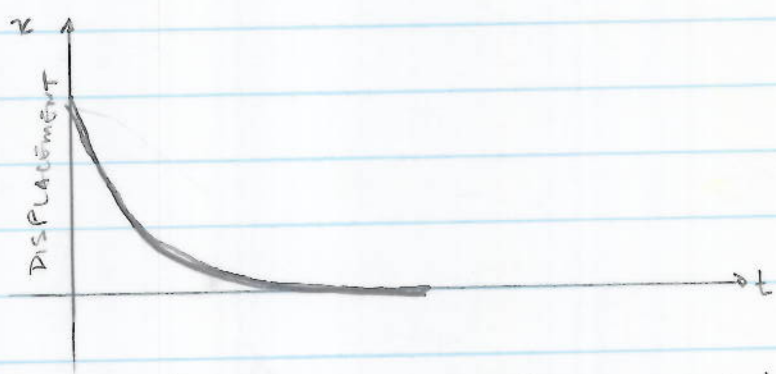
This CANNOT be a general solution since it contains only one constant. We can show that in such cases, if $e^{-\gamma t}$ is a solution, then

$$x = te^{-\gamma t} \qquad \text{a HW. (5.24)}$$

is also a solution

$$x(t) = (B_1 + B_2 t)e^{-\gamma t}$$

where B_1 and B_2 are constants to be determined by the initial conditions



critical damping plays a VERY IMPORTANT role for when it is desired that the system attain an equilibrium position rapidly and smoothly in the presence of frictional damping

CASE III Overdamped Oscillations

$$\gamma^2 > \omega_0^2$$

The two roots λ_1 and λ_2 are real. If we set

$$(\gamma^2 - \omega_0^2)^{1/2} = \omega_2$$

The general solution is

$$x(t) = e^{-\gamma t}\left(A_1 e^{\omega_2 t} + A_2 e^{-\omega_2 t}\right)$$

Note that ω_2 is no longer a frequency because the motion is no longer oscillatory. The exponents are real, and both terms on the right decay exponentially, one faster than the other.

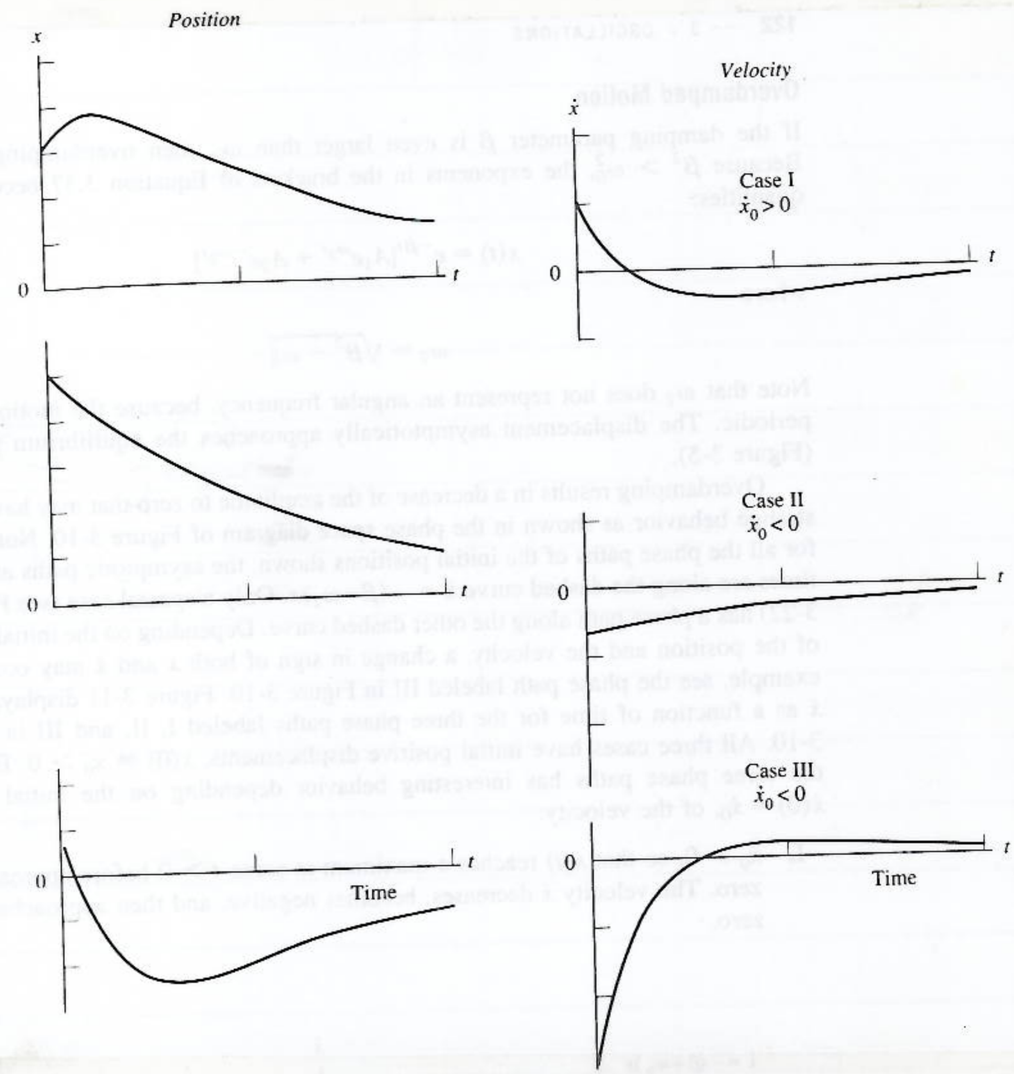
We can investigate three cases.

1. $\dot{x}_0 > 0$, so that $x(t)$ reaches a maximum at some $t > 0$ before approaching zero. The velocity $\dot{x}$ decreases, becomes negative, and then approaches zero

2. $\dot{x}_0 < 0$, with $x(t)$ and $\dot{x}(t)$ monotonically approaching $\emptyset$

3. $\dot{x} < 0$, but below the curve $\dot{x} = -(\gamma + \omega_2)x$, so that $x(t)$ goes negative before approaching zero, and $\dot{x}(t)$ goes positive before approaching zero

3-17



ENERGY CONSIDERATIONS

The total energy $E(t)$ of a damped harmonic system at any time t is given by:

$$E(t) = E(0) + W_f$$

$E(0)$ = the total energy at $t=0$

W_f : Work done by friction in the time interval between 0 and t.

3-18

Assuming the dissipative frictional force $f = -b\dot{x}$, we can calculate W_f as follows

$$W_f = \int f dx = \int f \frac{dx}{dt} dt = \int f v dt = -\int b v^2 dt$$

Thus the rate of energy loss by friction may be written as:

$$\frac{dE}{dt} = \frac{dW_f}{dt} = -bv^2.$$

which is a negative quantity and represents the rate at which energy is being dissipated into heat. Since $W_f < 0$, E_t continuously decreases with time and may calculated in the following manner:

$$E(t) = K(t) + U(t) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

For an underdamped harmonic oscillator

$$x(t) = Ae^{-\gamma t}\cos(\omega_1 t + \phi)$$

$$\dot{x}(t) = -\omega_1 A e^{-\gamma t}\left[\sin(\omega_1 t + \phi) + \frac{\gamma}{\omega_1}\cos(\omega_1 t + \phi)\right]$$

Let assume, furthermore, that the system is lightly damped so that $\gamma/\omega_1 \ll 1$, we may neglect the second term in $\dot{x}(t)$. We obtain

$$E(t) = \frac{1}{2}A^2 e^{-2\gamma t}\left[m\omega_1^2 \sin^2(\omega_1 t + \phi) + k\cos^2(\omega_1 t + \phi)\right]$$

3-19

Since we assumed light damping, we may write $\omega_1^2 \simeq \omega_0^2 = k/m$. E(t) then assumes the form

$$E(t) = \frac{1}{2} k A^2 e^{-2\gamma t} \quad \text{or} \quad E(t=0) = \frac{1}{2} k A^2$$

$$E(t) = E_0 e^{-2\gamma t}$$

Hence the energy decreases exponentially at a much faster rate ($e^{-2\gamma t}$) than the rate at which the amplitude decreases ($e^{-\gamma t}$)

Quality Factor

The quality factor Q is a dimensionless quantity and represents the degree of damping of an oscillator. The quality factor is defined as:

$$Q = 2\pi \frac{\text{energy stored in one cycle}}{\text{average energy dissipated in one cycle}}$$

If P is defined as the power loss or the rate at which the energy is dissipated, and the time period of oscillation is $T_1 = 2\pi/\omega_1$, we can write the denominator as

$$PT_1 = P \frac{2\pi}{\omega_1}$$

3-20

Therefore

$$Q = 2\pi \frac{E}{P\frac{2\pi}{\omega_1}} = \frac{E}{P/\omega_1}$$

But $1/\omega_1$ is the time of motion for 1 radian. We have therefore.

$$Q = \frac{\text{energy stored in the oscillator}}{\text{average energy dissipated per radian}}$$

This implies

<u>Q large</u> for lightly damped systems
<u>Q small</u> for heavily damped systems.

For a lightly damped system, we have derived

$$E(t) = E_0 e^{-2\gamma t}$$

and

$$\frac{dE}{dt} = -2\gamma E$$

The energy dissipated in time Δt will be

$$\Delta E = \left|\frac{dE}{dt}\right| \Delta t = 2\gamma E \Delta t$$

3-21

If Δt is the time for one radian of oscillation

$$\Delta t = \frac{1}{\omega_1}$$

$$Q = \frac{E}{\Delta E} = \frac{E}{2\gamma E/\omega_1} = \frac{\omega_1}{2\gamma}$$

For light damping $\omega_1 \cong \omega_0$

$$Q \simeq \frac{\omega_1}{2\gamma} \simeq \frac{\omega_0}{2\gamma}$$

Heavily damped systems such as rubberbands and loudspeakers have Q values in the range of 5 to 100. Systems such as tuning forks and violin strings may have a Q value as high as 1000. A typical microwave cavity resonator has a Q value of 10^4. And the rf cryomodules of CEBA @ JLab has a Q value of around 10^{10}. Moreover, systems with extremely light damping are: excited atoms ($Q \simeq 10^7$), excited nuclei ($Q \simeq 10^{12}$), and gas lasers ($Q \simeq 10^{14}$).

FORCED HARMONIC OSCILLATOR

A free oscillator will oscillate forever. In reality, however, every system has some damping present, and the system will eventually stop oscillating. To maintain the oscillations, energy from an external source must be supplied at a rate equal to the energy dissipated by the oscillator in the damping medium. Such motion in which energy is supplied externally is call FORCED OSCILLATIONS or a DRIVEN OSCILLATOR.

If a system is acted on by a driving force F_d then the net force F_{net} acting on the system is given by

$$F_{net} = F_s + F_f + F_d$$

where $F_s = -kx$ and $F_f = -b\dot{x}$

Let us assume that the driving force has a sinusoidal form

$$F_d = F_0 \cos(\omega t + \theta_0)$$

This is a reasonable characterization of the driving force, for by means of Fourier series, we can represent any periodic function of time as a sum of harmonic functions (to be discussed in a later lecture)

3-23

We may combine these above equations for F_s, F_f, and F_d and making use of Newton's Second Law:

$$\longrightarrow \quad m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega t + \theta_0) \qquad (*)$$

This is an inhomogeneous, second order, linear differential equation. The solution of which is given by the sum of two parts according to the following Thm.

<u>Theorem</u>: If $x_i(t)$ is a particular solution of an inhomogeneous differential equation and the complementary function $x_h(t)$ is a solution of the corresponding homogeneous differential equation [e.g. $m\ddot{x} + b\dot{x} + k\dot{x} = 0$], then $x(t) = x_i(t) + x_h(t)$ is also a solution of the inhomogeneous differential equation.

Thus the general solution of $(*)$ is

$$x(t) = x_i(t) + x_h(t)$$

where x_h is the solution of the homogeneous equation given by:

$$x_h(t) = e^{-\gamma t}\left[A_1 e^{+i\omega_1 t} + A_2 e^{-i\omega_1 t}\right]$$

$$x_h(t) = e^{-\gamma t}\left[B \sin\omega_1 t + C \cos\omega_1 t\right]$$

$$x_h(t) = A_h e^{-\gamma t} \cos(\omega_1 t + \phi_h).$$

3-24

Since the oscillations of a damped oscillator eventually decay to ZERO, the x_h part of the solution is called the <u>TRANSIENT TERM</u>. After a certain period of time, the x_h part of the solution is of no consequence; hence, for a steady-state solution we must concentate on finding the particular solution $x_i(t)$

According to equation (*), the applied force varies sinusoidally, so we can expect the resulting steady-state solution $x_i(t)$ to vary sinusoidally as well. Because of the $\dot{x}$ term on the left side of equation (*) we must introduce a phase. Let us assume a solution of the form

$$x_i(t) = A\cos(\omega t - \phi)$$

$$\dot{x}_i(t) = -A\omega \sin(\omega t - \phi)$$

$$\ddot{x}_i(t) = -A\omega^2 \cos(\omega t - \phi)$$

substituting these expressions into equation *:

$$-m\omega^2 A\cos(\omega t-\phi) - b\omega A\sin(\omega t-\phi) + kA\cos(\omega t-\phi)$$

$$= F_0 \cos\omega t$$

we have set $\theta_0 = 0$

3-25

Rearranging,

$$(kA\cos\phi - m\omega^2 A\cos\phi + b\omega A\sin\phi)\cos\omega t$$

$$-(kA\sin\phi - m\omega^2 A\sin\phi - b\omega A\cos\phi)\sin\omega t$$

$$= F_0\cos\omega t$$

Because $\sin\omega t$ and $\cos\omega t$ are linearly-independent functions, the coefficients of $\sin\omega t$ and $\cos\omega t$ must be separately equal. That is:

(1) $$(k - m\omega^2)\cos\phi + b\omega\sin\phi = \frac{F_0}{A}$$

(2) $$(k - m\omega^2)\sin\phi - b\omega\cos\phi = 0$$

From (2), we obtain an expression for the phase angle

$$\tan\phi = \frac{b\omega}{k - m\omega^2} = \frac{b\omega/m}{k/m - \omega^2}$$

Using the usual notation $k/m = \omega_0^2$ & $\gamma = b/2m$ we obtain

$$\tan\phi = \frac{2\gamma\omega}{\omega_0^2 - \omega^2}$$

And we can further obtain:

$$\sin\phi = \frac{2\gamma\omega}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}$$

and

$$\cos\phi = \frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}$$

If we substitute the expressions into equation (1) above, we obtain.

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}$$

Thus a particular solution of the inhomogeneous equation is (STEADY-STATE SOLUTION)

!

$$x_i(t) = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)+4\gamma^2\omega^2}}\cos(\omega t-\phi)$$

where $$\phi = \tan^{-1}\frac{2\gamma\omega}{\omega_0^2-\omega^2}$$

3-27

The general solution is the combination of the homogeneous and the inhomogeneous solutions.

$$x = x_h + x_i$$

$$= A_h e^{-\gamma t} \cos(\omega_1 t + \phi_h)$$

$$+ \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \cos(\omega t - \phi)$$

As required, this solution contains two arbitrary constants (of integration) A_h & ϕ_h, while ϕ is not a constant as is given above. The first part of the solution oscillates with a natural frequency ω_1. Because of the damping, the oscillations die out for large values of time, that is, for $t \gg 1/\gamma$. The homogeneous solution x_h is called the transient solution, while the particular solution x_i is the steady-state solution.

Since for $t \gg 1/\gamma$ $x \simeq x_i$, we shall concentrate on the discussion of the steady-state solution. This solution is INDEPENDENT of the INITIAL CONDITIONS.

AMPLITUDE RESONANCE

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}$$

$$\phi = \tan^{-1} \frac{2\gamma\omega}{\omega_0^2 - \omega^2}$$

We observe that for a fixed value of ω_0, the variations in A and ϕ are strongly dependent upon the ratio of $\frac{\omega}{\omega_0}$

Depending upon the value of γ, there are certain driving frequencies at which the amplitude A has a maximum value. The frequency at which the amplitude has a maximum is called the

* AMPLITUDE RESONANCE FREQUENCY ω_r *

This frequency ω_r may be calculated by setting

$$\left.\frac{dA}{d\omega}\right|_{\omega = \omega_r} = 0$$

Upon solving the resulting equation, we get

$$\omega = \omega_r = (\omega_0^2 - 2\gamma^2)^{1/2}$$

In the case of small damping

$$\omega_r = \omega_0 \left(1 - \frac{1}{2}\left(\frac{2\gamma^2}{\omega_0^2}\right) + \cdots \right)$$

$$\omega_r \simeq \omega_0 - \frac{\gamma^2}{\omega_0}$$

Thus the maximum amplitude $A = A_0$ occurs at $\omega = \omega_r$

$$A_0 = \frac{F_0/m}{2\gamma\left(\sqrt{\omega_0^2 - \gamma^2}\right)}$$

In the case of small damping $\gamma \to 0$

$$A_0 \simeq \frac{F_0}{2m\gamma\omega_0} = \frac{F_0}{b\omega_0}$$

It is clear then, if b is small, the amplitude A_0 can become VERY LARGE.

ENERGY RESONANCE

In most practical situations involving oscillations in nature, the quantity observed experimentally is the ENERGY not the AMPLITUDE

For steady-state motion, the amplitude A is constant

$$x = A\cos(\omega t + \phi)$$

$$v = \dot{x} = -\omega A \sin(\omega t + \phi)$$

3-30

It then follows.

$$K(t) = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi)$$

$$U(t) = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t + \phi)$$

The mechanical energy $E(t) = K(t) + U(t)$ is:

$$E(t) = \frac{1}{2}A^2\left[m\omega^2\sin^2(\omega t + \phi) + k\cos^2(\omega t + \phi)\right]$$

Now let us calculate the time averages of $K(t)$, $U(t)$ and $E(t)$ in the case when A changes with ω, i.e, $A = A(\omega)$

Again, $$A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2) + 4\gamma^2\omega^2}}$$

Therefore we may write

$$K(t) = \frac{1}{2}m\omega^2\left(\frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2}}\right)^2 \sin^2(\omega t + \phi)$$

Now since $\langle \sin^2(\omega t + \phi)\rangle = \frac{1}{2} = \langle \cos^2(\omega t + \phi)\rangle$
(for an average over one period)

$$\langle K(t)\rangle = \frac{1}{4}\frac{F_0^2}{m}\frac{\omega^2}{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2}$$

3-31

The value of ω for which $\langle K \rangle$ is maximum is obtained by setting

$$\left.\frac{d\langle K \rangle}{d\omega}\right|_{\omega=\omega_K} = 0$$

which gives

$$\boxed{\omega_K = \omega_0}$$

$\Rightarrow$ That is, the kinetic Energy resonance occurs at the natural free frequency ω_0 $\Leftarrow$

Since $U(t) = \frac{1}{2}kx^2$, the potential energy resonance must occur at the same position as the amplitude resonance.

The potential energy resonance frequency ω_U is:

$$\boxed{\omega_U = \omega_r = (\omega_0^2 - 2\gamma^2)^{1/2}}$$

It should not alarm you unduly to find that the kinetic and potential energies resonate at different frequencies. Because we are dealing with a damped oscillator - which is a nonconservative system - the energy is continuously being drained from the system

3-32

The average mechanical energy - averaged over one period is

$$\langle E(t)\rangle = \frac{1}{2}A^2 m\omega^2\langle \sin^2(\omega t+\phi)\rangle + \frac{1}{2}kA^2\langle\cos^2(\omega t+\phi)\rangle$$

$$= \cancel{\frac{1}{4}A^2[m\omega^2+\omega_0^2]} = \frac{1}{4}mA^2[\omega^2+\omega_0^2]$$

Substituting for A gives:

$$\langle E\rangle = \frac{1}{4}\frac{F_0^2}{m}\frac{\omega^2+\omega_0^2}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}$$

For very weak damping: $\gamma^2 \ll \omega_0^2$ and $\omega \simeq \omega_0$

$$\omega^2+\omega_0^2 \cong 2\omega_0^2$$

Let $\omega = 10$, $\omega_0 = 9$ $\omega^2-\omega_0^2 \simeq 20$

and

$$\omega^2-\omega_0^2 = (\omega+\omega_0)(\omega-\omega_0) \simeq 2\omega_0(\omega-\omega_0)$$

Applying these approximations and observing that $E = E(\omega)$, we can replace $\langle E(t)\rangle$ by $\langle E(\omega)\rangle$ and simplify

$$\langle E(\omega)\rangle = \frac{1}{4}\frac{F_0^2}{m}\frac{2\omega_0^2}{(2\omega_0(\omega-\omega_0))^2+4\gamma^2\omega_0^2}$$

$$= \frac{1}{8}\frac{F_0^2}{m}\frac{1}{(\omega-\omega_0)^2+\gamma^2}$$

We can recast this relation:

$$\langle E(\omega)\rangle \frac{8m}{F_0^2} = \frac{1}{(\omega-\omega_0)^2 + \gamma^2} = L(\omega)$$

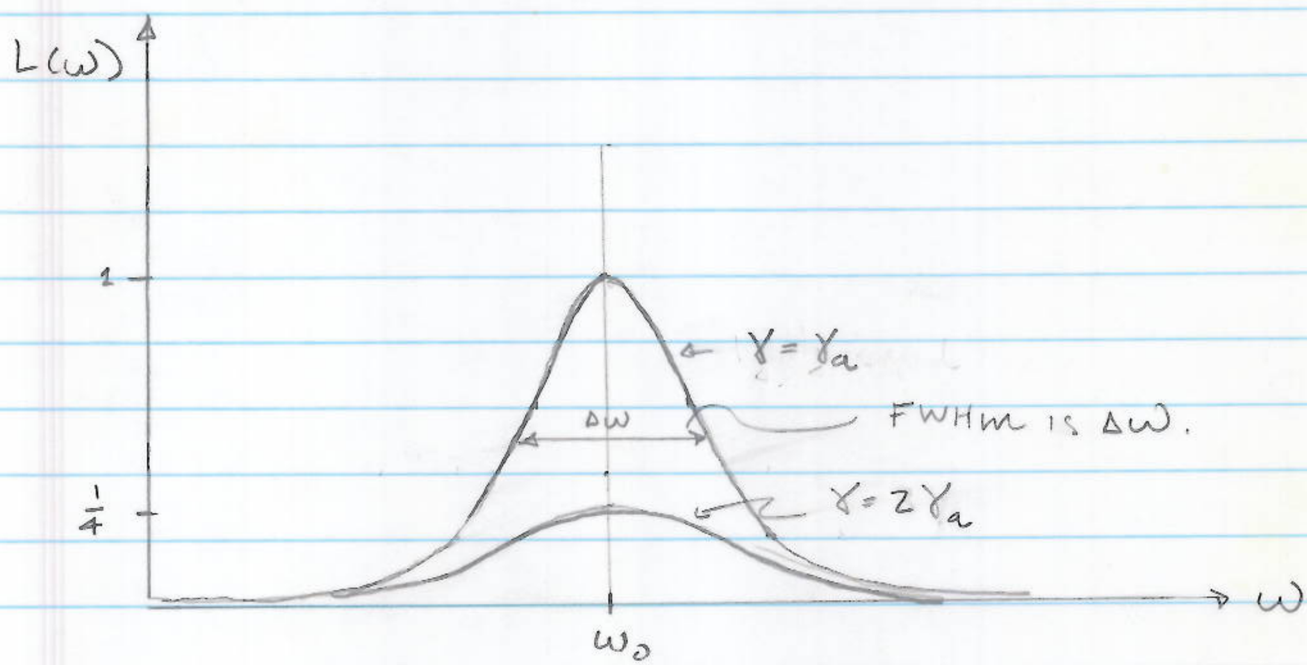
where the function $L(\omega)$ contains all the necessary frequency dependence of $E(\omega)$.

A plot of the function $L(\omega)$ is called a

RESONANCE CURVE

or

LORENTZIAN



Note that for large γ the function is effectively zero except for near the resonance frequency ω_0.

3-34

The maximum height of the resonance curve occurs at $\omega = \omega_0$ and equals $L(\omega=\omega_0) = \frac{1}{\gamma^2}$

This value will fall to one-half of its maximum value when

$$(\omega-\omega_0)^2 = \gamma^2$$

or

$$(\omega-\omega_0) = \pm\gamma$$

This equation states that the resonance curve drops to half its maximum value at $\omega_+ = \omega_0+\gamma$ on the higher frequency side of ω_0 and at $\omega_- = \omega_0-\gamma$ on the lower frequency side of ω_0.

The Full Width Half Maximum (FWHM) of the resonance curve is

$$\Delta\omega = 2\gamma$$

As γ decreases, the width $\Delta\omega$ of the curve also decreases. $\Rightarrow$ This means as $\gamma \to 0$ the Lorentzian becomes <u>SHARPER</u>, $\Rightarrow$ The oscillating system becomes increasingly more SELECTIVE in frequency.

The frequency-selective property of an oscillating system is characterized by the

Quality Factor Q.

Earlier we defined (and showed)

$$Q = \frac{\text{Energy stored in the oscillator}}{\text{Energy Dissipated per radian of oscillation}}$$

For a lightly-damped oscillator, we showed that

$$Q \cong \frac{\omega_0}{2\gamma}$$

When such an oscillator is driven, we get a resonance width of $\Delta\omega = 2\gamma$. We may now write:

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\text{resonance frequency.}}{\text{frequency width of resonance curve}}$$

⇒ Systems with a high value of Q have a very narrow resonance width and hence are very selective to a frequency response when external driving forces are applied. If the resonance is sharp, a system will respond only when the driving frequency is equal to the resonance frequency

Rate of Energy Dissipation

Let us calculate the rate at which energy is being dissipated. This should equal the rate at which work is being done. Starting with the general equation for a forced oscillator, with $\theta_0 = 0$

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos\omega t$$

multiplying both sides by $\dot{x}$

$$m\ddot{x}\dot{x} + b\dot{x}^2 + kx\dot{x} = F_0 \dot{x} \cos\omega t$$

And we may write this as:

$$\frac{d}{dt}\left(\frac{m\dot{x}^2}{2} + \frac{m\omega_0^2 x^2}{2}\right) + b\dot{x}^2 = (F_0 \cos\omega t)\dot{x}$$

or

$$\frac{d}{dt}(K + U) + b\dot{x}^2 = (F_0 \cos\omega t)\dot{x}$$

we found earlier that the rate of energy loss by friction is $-bv^2$. We observe that the term on the RHS of the above equation is the power

3-37

$$\begin{pmatrix}\text{Time rate}\\ \text{of change of}\\ K \text{ and } U\end{pmatrix} = \begin{pmatrix}\text{Rate at which}\\ \text{energy is being}\\ \text{dissipated}\end{pmatrix} + \begin{pmatrix}\text{Rate at which energy}\\ \text{is being supplied}\\ \text{by the driving force}\end{pmatrix}$$

After the transients have died out

$$\langle E \rangle = \langle b\dot{x}^2 \rangle$$

$$\Rightarrow \langle E \rangle_{\substack{\text{one}\\\text{cycle}}} = b\int_0^T \dot{x}^2 dt = \left[b\,\frac{1}{T}\int_0^T \dot{x}^2 dt\right]T$$

If $x = A\cos(\omega t + \phi)$ $\qquad T = \dfrac{2\pi}{\omega}$

$$\dot{x} = -A\omega\sin(\omega t + \phi)$$

$$\langle E \rangle = \left[b\,\frac{1}{T}\int_0^T A^2\omega^2\sin^2(\omega t+\phi)\,dt\right]T$$

$$= A^2\omega^2 b\,T\,\underbrace{\left[\frac{1}{T}\int_0^T \sin^2(\omega t+\phi)\,dt\right]}_{\to\ \frac{1}{2}}$$

$$= 2\pi\omega A^2 b$$

$$= 2\pi\omega A^2(2m\gamma) = 2\pi m\omega\gamma A^2.$$