$\boxed{\text{Principle of Superposition}}$ & Fourier Series.

→ When two or more waves travel simultaneously through a portion of a medium, each wave acts INDEPENDENTLY as if the other waves were not present. The resultant displacement at any point is the vector sum of the displacements of the individual waves.

The second-order linear differential equation describing a forced harmonic oscillator is given by:

$$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = F(t)$$

This equation may be written in the general form.

$$\left(\frac{d^2}{dt^2} + a\frac{d}{dt} + b\right) x(t) = F(t)$$

We define a linear operator, $\mathbb{L}$, as the quantity in the parentheses on the lhs of the equation:

$$\mathbb{L} = \frac{d}{dt^2} + a\frac{d}{dt} + b$$

so that

$$\mathbb{L}\, x(t) = F(t)$$

Linear Operators obey the principle of superposition. This property results from the fact that linear operators are distributive.

$$\mathbb{L}(x_1 + x_2) = \mathbb{L}x_1 + \mathbb{L}x_2$$

Hence, if we have two solutions $x_1(t)$ and $x_2(t)$ for two forcing functions $F_1(t)$ and $F_2(t)$

$$\mathbb{L}x_1 = F_1(t) \quad \& \quad \mathbb{L}x_2 = F_2(t)$$

we can add these equations (multiplied by arbitrary constants α_1 & α_2) and obtain

$$\mathbb{L}(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 F_1(t) + \alpha_2 F_2(t)$$

We can extend this argument to a set of solutions $x_n(t)$, each of which is appropriate for a given $F_n(t)$

$$\mathbb{L}\left(\sum_{n=1}^{N} \alpha_n x_n(t)\right) = \sum_{n=1}^{N} \alpha_n F_n(t)$$

IF we collapse the linear combinations:

$$x(t) = \sum_{n=1}^{N} \alpha_n x_n(t)$$

$$F(t) = \sum_{n=1}^{N} \alpha_n F_n(t)$$

Then we have

$$\mathbb{L}\, x(t) = F(t)$$

Proof: $\mathbb{L}\, x(t) = \mathbb{L}\left(\sum_n \alpha_n x_n(t) \right)$

$$= \sum \alpha_n \mathbb{L}\, x_n(t) = \sum \alpha_n F_n(t) = F(t)$$

As required.

If each of the individual functions $F_n(t)$ has a simple harmonic dependence on time, such as $\cos(\omega_n t + \phi_n)$, we know the corresponding solution $x_n(t)$ is given

$$x_n(t) = A e^{-\gamma t} \cos(\omega_1 t + \theta') + \frac{F_n}{m} \frac{\cos(\omega_n t - \phi_n)}{\sqrt{(\omega_0^2 - \omega_n^2)^2 + 4\gamma^2 \omega_n^2}}$$

Now imagine we have N Forces with a harmonic dependence on time. The general solution will be a linear combination of these forces:

$$x(t) = A e^{-\gamma t} \cos(\omega_1 t + \theta') + \sum_{n=1}^{N} \frac{F_n}{m} \frac{\cos(\omega_n t - \theta_n - \phi_n)}{\sqrt{(\omega_0^2 - \omega_n^2)^2 + 4\gamma^2 \omega_n^2}}$$

where

$$\phi_n = \tan^{-1}\left(\frac{2\gamma\omega_n}{\omega_0^2-\omega_n^2}\right)$$

We can write down similar solutions where $F(t)$ is represented by a series of terms – $\sin(\omega_n t - \theta_n)$

With the help of the FOURIER Theorem, we can extend this argument to any driving force which is

1) PERIODIC $(F(t+T) = F(t))$; $T = \frac{2\pi}{\omega}$

2) CONTINUOUS or PIECEWISE CONTINUOUS

<u>Fourier Thm</u> Any arbitrary periodic function, which is continuous or piecewise continuous, having only a finite number of discontinuities over a time period, can be expressed a sum of harmonic terms!

* Thus any function $F(t)$ that is defined within a time interval $-T/2 < t < T/2$ can be expressed as a series of sine and cosine terms:

$$F(t) = \frac{a_0}{2} + a_1\cos\omega t + a_2\cos 2\omega t + \dots + a_n\cos n\omega t$$
$$+ b_1\sin\omega t + b_2\sin 2\omega t + \dots + b_n\sin n\omega t$$

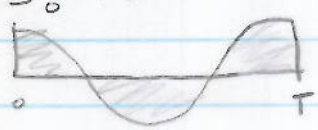
And this can be written as

$$F(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$

we observe

$$\int_0^T F(t)\cos m\omega t\,dt =$$

$$\cancelto{0}{\int_0^T \frac{1}{2}a_0 \cos m\omega t} + \sum_{n=1}^{\infty}\int_0^T (a_n \cos n\omega t \cos m\omega t + b_n \sin n\omega t \cos m\omega t)\,dt$$



$$\int_0^T \cos n\omega t \cos m\omega t\,dt = \begin{cases} T/2 & n = m \\ 0 & n \neq m \end{cases}$$

$$\int_0^T \sin n\omega t \cos m\omega t\,dt = 0 \quad \text{for all integral } m \text{ and } n$$

Hence in the expression above only the $m=n$ term survives and.

$$\int_0^T F(t)\cos n\omega t\,dt = a_n \cancelto{T/2}{\int_0^T \cos^2 n\omega t\,dt}$$

$\Rightarrow$

$$a_n = \frac{2}{T}\int_0^T F(t)\cos n\omega t\,dt$$

Likewise

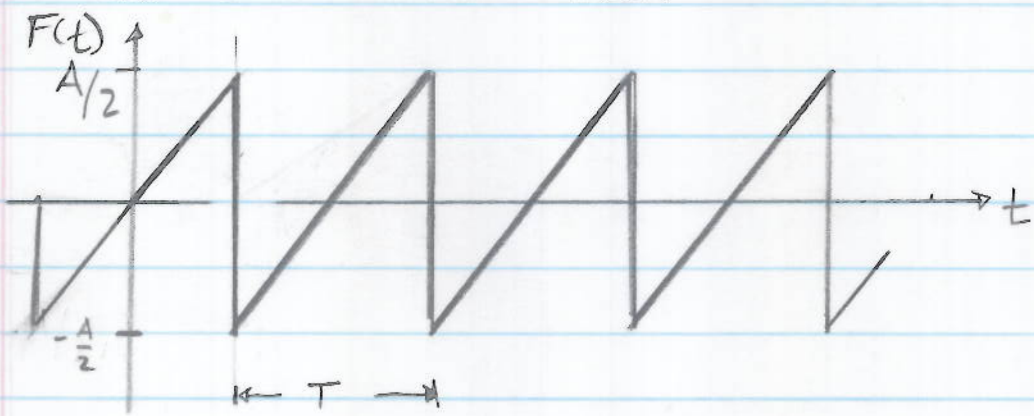
$$b_n = \frac{2}{T}\int_0^T F(t)\sin n\omega t\,dt$$

Because $F(t)$ has a period of T, we can replace the integral limits 0 and T by the limits $-\frac{1}{2}T = -\frac{\pi}{\omega}$ and $+\frac{1}{2}T = +\frac{\pi}{\omega}$:

$$a_n = \frac{\omega}{\pi}\int_{-\pi/\omega}^{+\pi/\omega} F(t)\cos n\omega t\, dt$$

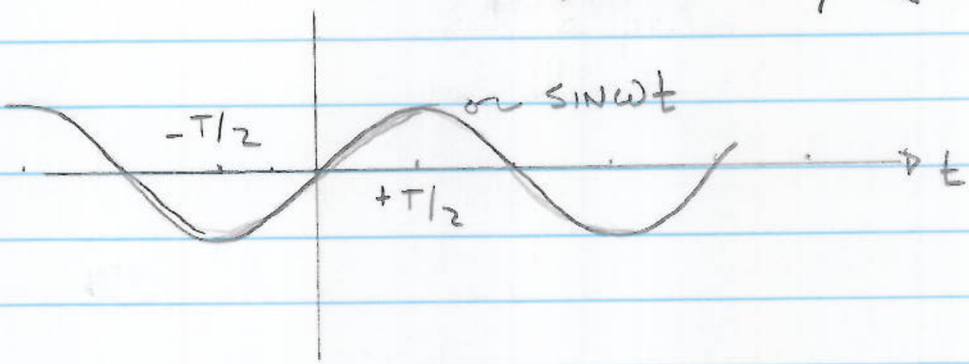
$$b_n = \frac{\omega}{\pi}\int_{-\pi/\omega}^{+\pi/\omega} F(t)\sin n\omega t\, dt$$

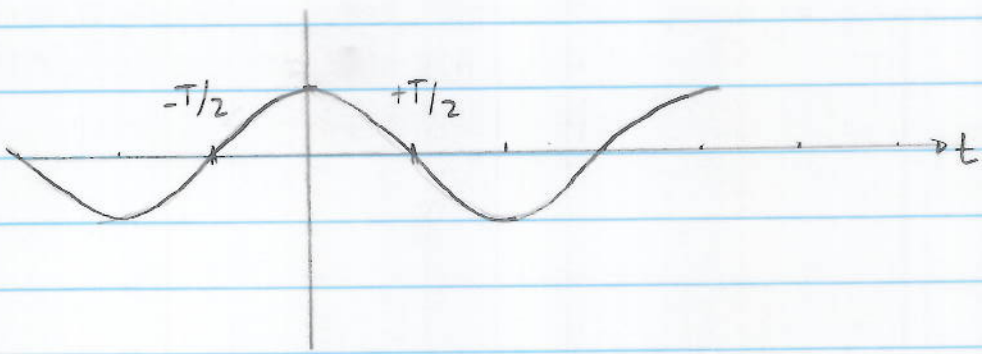
<u>Example</u> A sawtooth driving force function is shown below. Find the coefficients a_n and b_n, and express $F(t)$ as a Fourier series.



$$F(T) = A\frac{t}{T} = \frac{\omega A}{2\pi}t \qquad -\frac{T}{2} < t < \frac{T}{2}$$

Now let's invoke the properties of symmetry





over the interval $-\frac{T}{2}$ to $+\frac{T}{2}$:

SIN $n\omega t$ IS an ODD function of $n\omega t$
COS $n\omega t$ IS an EVEN function of $n\omega t$

If $F(t)$ is an even function over the interval of $-T/2 < t < +T/2$, then all $b_n = 0$. Likewise if $F(t)$ is an odd function over the interval of $-T/2 < t < +T/2$, then all $a_n = 0$.

For the sawtooth function $F(t) = A\frac{t}{T}$: we see it is an ODD function. Hence $a_n = 0$ (i.e. all coefficients a_n vanish identically)

The b_n are given by:

$$b_n = \frac{\omega}{\pi}\int_{-\pi/\omega}^{+\pi/\omega} F(t)\sin n\omega t\,dt$$

$$= \frac{\omega^2 A}{2\pi^2}\int_{-\pi/\omega}^{+\pi/\omega} t\sin n\omega t\,dt$$

$$= \frac{\omega^2 A}{2\pi^2}\left[-\frac{t\cos n\omega t}{n\omega} + \underbrace{\frac{\sin n\omega t}{n^2\omega^2}}_{\emptyset}\right|_{-\pi/\omega}^{+\pi/\omega}$$

$$= +\frac{A}{n\pi} \quad (n \text{ odd})$$

$$= -\frac{A}{n\pi} \quad (n \text{ even})$$

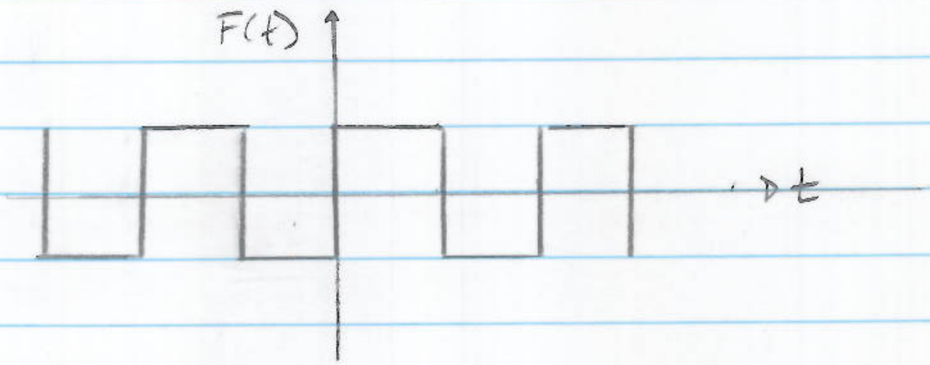
$$\Rightarrow b_n = \frac{A}{n\pi}(-1)^{n+1}$$

$$F(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t$$

$$= \frac{A}{n\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\sin n\omega t$$

Example SQUARE WAVE

Let $F(t) = \begin{cases} -1 & -T/2 < t < 0 \\ +1 & 0 < t < +T/2 \end{cases}$



F(t) is an odd function of t so all coefficients a_n vanish identically.

$$b_n = \frac{\omega}{\pi}\int_{-\pi/\omega}^{+\pi/\omega} F(t)\,dt$$

$$= -\frac{\omega}{\pi}\int_{-\pi/\omega}^{0} \sin n\omega t\,dt + \frac{\omega}{\pi}\int_{0}^{+\pi/\omega} \sin n\omega t\,dt$$

$$b_n = \begin{cases} \dfrac{4}{n\pi} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$

$$\Rightarrow F(t) = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\sin[(2n+1)\omega t]}{(2n+1)}$$

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or

$$F(t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin[(2n-1)\omega t]}{(2n-1)}$$

Now let's integrate $F(t)$ over the region $[0, \pi/\omega]$

$$\underbrace{\int_0^{\pi/\omega} F(t)\,dt}_{=\pi/\omega} = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)} \int_0^{\pi/\omega} \sin[(2n-1)\omega t]\,dt$$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)} \left[\frac{1}{(2n-1)\omega}\left(-\cos[(2n-1)\omega t]\right]\right|_0^{\pi/\omega}$$

$$= \frac{4}{\pi}\sum \frac{2}{\omega(2n-1)^2} = \frac{8}{\omega\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

And we have a neat way for calculating π

$$\pi = \left[8\sum \frac{1}{(2n-1)^2}\right]^{1/2}$$

Nonlinear Oscillating Systems.

Let us now consider systems in which the restoring force is not proportional to the displacement. In general:

$$m\ddot{x} + F(x) = 0$$

F(x) is the restoring force and is NOT LINEAR.

If damping is present, we shall have another free function $G(\dot{x})$, which may also be NONLINEAR

One OUTSTANDING characteristic of a nonlinear system is that, UNLIKE LINEAR SYSTEMS,

$$T = T(A) \quad \& \quad \omega = \omega(A)$$

The time period of oscillation depends upon the amplitude.

Consider a system displaced to a position x from its equilibrium position x_0 and is under a restoring force $F(x)$. Let us expand $F(x)$ in a Taylor series about x_0

$$F(x) = F(x_0) + \left(\frac{dF}{dx}\right)_{x=x_0}(x-x_0) + \frac{1}{2}\left(\frac{d^2F}{dx^2}\right)_{x=x_0}(x-x_0)^2$$

$$+ \frac{1}{6}\left(\frac{d^3F}{dx^3}\right)_{x=x_0}(x-x_0)^3 + \cdots$$

Here $F(x_0) = 0$ because x_0 is the equilibrium point

We shall define

$$\left(\frac{dF}{dx}\right)_{x=x_0} = k_1 \,,\ \frac{1}{2}\left(\frac{d^2F}{dx^2}\right)_{x=x_0} = k_2 \,,\ \frac{1}{6}\left(\frac{d^3F}{dx^3}\right)_{x=x_0} = k_3$$

$F(x)$ then becomes.

$$F(x) = k_1(x-x_0) + k_2(x-x_0)^2 + k_3(x-x_0)^3 + \cdots$$

If we consider only those forces that lead to stable equilibrium for symmetrical systems, the even terms must vanish. that is $k_2 = k_4 = k_6 = \cdots = 0$ and to leading order

$$F(x) = k_1 x + k_3 x^3 \quad \text{+ we have set } x_0 = 0$$

This force is symmetrical about $x = 0$, the equilibrium point. This means the magnitude of the force exerted upon the system is the same for x and $-x$. If we set

$$k_1 = -k \quad \text{and} \quad k_3 = +\epsilon$$

we get

$$F(x) \cong -kx + \epsilon x^3$$

if $\epsilon > 0$, the force is less than the linear term alone and is said to be SOFT

if $\epsilon < 0$, the force is greater than the linear term alone and is said to be HARD

The potential corresponding to such a force is

$$U(x) = \frac{1}{2}kx^2 - \frac{1}{4}\epsilon x^4$$

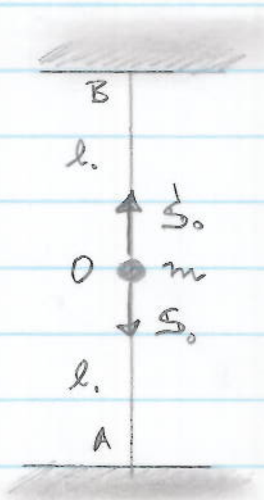
On the other hand, if the system is UNSYMMETRICAL, and after substituting $k_1 = -k$ and $k_2 = -\lambda$ and setting $k_3 = 0$, we obtain for $F(x)$

$$F(x) = -kx - \lambda x^2$$

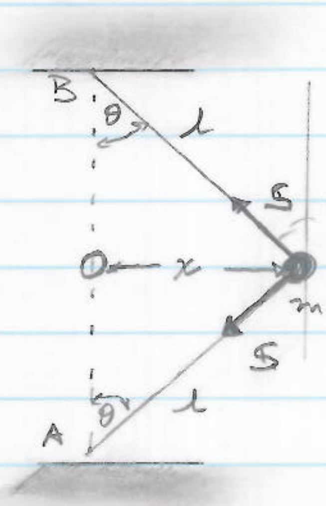
Hence the ASYMMETRICAL POTENTIAL IS:

$$U(x) = \frac{1}{2}kx^2 + \frac{1}{3}\lambda x^3$$

SYMMETRICAL NONLINEAR SYSTEM



(a) Equilibrium position



(b) Displaced position

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Consider a mass m which is suspended between two identical strings (or springs) as depicted in the figure above. The strings are elastic with a force constant k_o and tied to points A & B. When this system is in position AOB it is in equilibrium and the tension in each string is S_o. Let us now displace the mass m horizontally through a distance x. The change in the length of the string is $l-l_o$

$$\Delta l = l - l_o$$

The restoring force is (along the direction of the string)

$$|F| = k_o \Delta l = k_o(l-l_o)$$

The tension S in each string when it is in the displaced position is

$$S = S_o + k_o(l-l_o)$$

We resolve S into components. The vertical components cancel each other. The horizontal components sum to:

$$-2S\sin\theta$$

Thus the equation of motion is

$$m\ddot{x} = -2S\sin\theta$$

Here we have assumed that there is no damping force

$$m\ddot{x} = -2\left[S_0 + k_0(l - l_0)\right]\sin\theta$$

$$l = (l_0^2 + x^2)^{1/2} = l_0\left(1 + \frac{x^2}{l_0^2}\right)^{1/2}$$

and

$$\sin\theta = \frac{x}{l} = x(l_0^2 + x^2)^{-1/2} = \frac{x}{l_0}\left(1 + \frac{x^2}{l_0^2}\right)^{-1/2}$$

we have then.

$$m\ddot{x} = -2\left\{S_0 + k_0 l_0\left[\left(1 + \frac{x^2}{l_0^2}\right)^{1/2} - 1\right]\right\}\frac{x}{l_0}\left(1 + \frac{x^2}{l_0^2}\right)^{-1/2}$$

we shall assume that $\frac{x^2}{l_0^2}$ is small. Expanding by means of the binomial theorem.

$$\left(1 + \frac{x^2}{l_0^2}\right)^{\pm 1/2} \cong 1 \pm \frac{1}{2}\left(\frac{x^2}{l_0^2}\right) + \ldots$$

$$m\ddot{x} \cong -2\left\{S_0 + k_0 l_0\left(\frac{1}{2}\frac{x^2}{l_0^2}\right)\right\}\frac{x}{l_0}\left(1 - \frac{1}{2}\frac{x^2}{l_0^2}\right)$$

$$\Rightarrow m\ddot{x} = -\frac{2S_0}{l_0}x + \left(\frac{S_0}{l_0^3} - \frac{k_0}{l_0^2}\right)x^3 + O(x^5)$$

we shall let

$$k = \frac{2S_0}{l_0} \quad \text{and} \quad \epsilon = \left(\frac{S_0 - k_0 l}{l_0^3}\right)$$

Hence we may write

$$m\ddot{x} = -kx + \epsilon x^3 \qquad (*)$$

Note that ϵ is a small quantity that is positive for a soft spring and negative for a hard spring.

Let us assume that the approximate solution of $(*)$ is sinusoidal. This should be approximately true since ϵ is a small quantity.

Let $x = A\cos\omega t$

Substituting $x = x_1$ and $x^3 = x_1^3$

$$m\ddot{x}_1 = -kA\cos\omega t + \epsilon A^3\cos^3\omega t$$

Substituting

$$\cos^3\omega t = \frac{1}{4}(3\cos\omega t + \cos 3\omega t)$$

and rearranging we obtain

$$m\ddot{x}_1 = -\left(kA - \frac{3}{4}\epsilon A^3\right)\cos\omega t + \frac{1}{4}\epsilon A^3\cos 3\omega t$$

Upon integration (with the constants of integration set to zero) we have

$$x_1 = \frac{1}{m\omega^2}\left(kA - \frac{3}{4}\epsilon A^3\right)\cos\omega t - \frac{1}{36}\frac{\epsilon A^3}{m\omega^2}\cos 3\omega t$$

This is the solution for a FIRST-ORDER APPROXIMATION

To find the relationship between ω and A we can make use of the assumption that ϵ is small. Thus substituting a first-order approximation

$$x = x_1 = A\cos\omega t$$

and dropping the last term, we get

$$A = \frac{1}{m\omega^2}\left(kA - \frac{3}{4}\epsilon A^3\right)$$

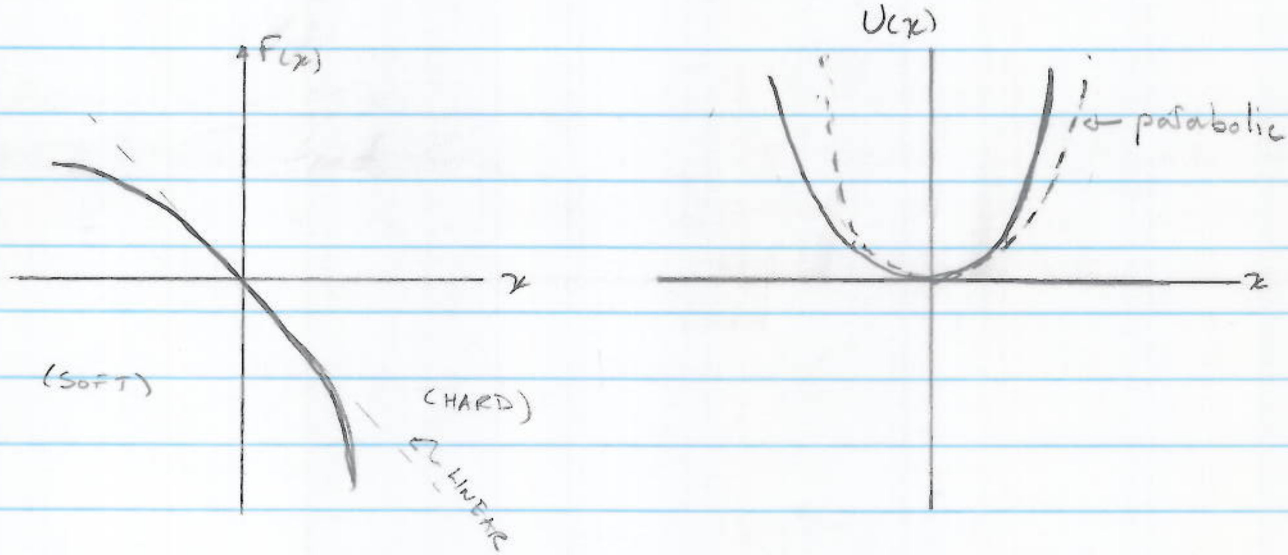
$$\Rightarrow \omega^2 = \frac{k}{m} - \frac{3}{4}\frac{\epsilon}{m}A^2$$

This relation indicates that the natural frequency ω and hence the period $T = 2\pi/\omega$ are functions of the amplitude A. The quantity ω^2 increases or decreases from ω_0^2 by an amount of $\frac{3}{4}\frac{\epsilon}{m}A^2$ depending on the magnitude and sign of ϵ.

Asymmetrical Nonlinear System

$F(x) = -kx - \lambda x^2$

$U(x) = \frac{1}{2}kx^2 + \frac{1}{3}\lambda x^3$



$\boxed{\lambda > 0}$ The system is hard for $x>0$ and is soft for $x<0$

The differential equation describing such a system without damping is:

$$m\ddot{x} + kx + \lambda x^2 = 0$$

Dividing by m and setting $\omega_0^2 = \frac{k}{m}$ & $\lambda_1 = \frac{\lambda}{m}$:

$$\ddot{x} + \omega_0^2 x + \lambda_1 x^2 = 0$$

If there were no nonlinear term ($\lambda_1 x^2$), the solution would be $x_0 = A\sin\omega_0 t$. Since λ_1 is a small term, the solution can be obtained by adding a small correction term to x_0

$$x(t) \cong x_0 + \lambda_1 x_1$$

[To have higher order corrections we must write
$$x(t) = x_0 + \lambda_1 x_1 + \lambda_1^2 x_2 + \lambda_1^3 x_3 + \cdots$$]

Substituting our approximate solution into our asymmetrical nonlinear 2nd order differential equation we get after some rearranging.

$$(\ddot{x}_0 + \omega_0^2 x_0) + (\ddot{x}_1 + \omega_0^2 x_1 + x_0^2)\lambda_1 + 2x_0 x_1 \lambda_1^2 + x_1^2 \lambda_1^3 = 0$$

Neglecting higher order terms in λ_1^2 and λ_1^3 results in:

$$(\ddot{x}_0 + \omega_0^2 x) + (\ddot{x}_1 + \omega_0^2 x_1 + x_0^2)\lambda_1 = 0$$

For this equation to be valid for any value λ_1, each term must be zero.

$$\ddot{x}_0 + \omega_0^2 \dot{x}_0 = 0$$

and

$$\ddot{x}_1 + \omega_0^2 x_1 + x_0^2 = 0$$

Notice that x_0 & x_1 are coupled.

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Suppose the initial conditions are such that

$$x_0 = A \sin \omega_0 t$$

Then

$$\ddot{x}_1 + \omega_0^2 x_1 = -x_0^2 = -A^2 \sin^2 \omega_0 t = -\frac{A^2}{2}(1 - \cos 2\omega_0 t)$$

The general solution to the differential equation is.

$$x_1(t) = B \cos 2\omega_0 t + C$$

Substituting into the above differential equation and rearranging:

$$-\left(\frac{A^2}{2} + 3\omega_0^2 B\right)\cos 2\omega_0 t - \left(\frac{A^2}{2} + \omega_0^2 C\right) = 0$$

For this to be true for any value of t we must have

$$+\frac{A^2}{2} + 3\omega_0^2 B = 0 \quad \Rightarrow \quad B = -\frac{A^2}{6\omega_0^2}$$

$$+\frac{A^2}{2} + \omega_0^2 C = 0 \quad \Rightarrow \quad C = -\frac{A^2}{2\omega_0^2}$$

$$\Rightarrow \quad x_1(t) = -\frac{A^2}{6\omega_0^2}\cos 2\omega_0 t - \frac{A^2}{2\omega_0^2}$$

Thus, the general steady-state solution for a first-order approximation in λ ($\lambda_1 = \frac{\lambda}{m}$) is

$$x(t) = x_0 + \lambda x_1 = A \sin \omega_0 t + \frac{\lambda A^2}{6 m \omega_0^2} (\cos 2\omega_0 t + 3)$$

We see that the solution contains not only the free natural frequency ω_0, but also the higher harmonic $2\omega_0$.

constant

3.25 $\ddot{\theta}+\sin\theta=0 \quad \Rightarrow \quad \frac{d}{dt}\left\{\frac{\dot{\theta}^2}{2}-\cos\theta\right\}=0$

Integrating: $\left.\frac{\dot{\theta}^2}{2}\right|_0^{\dot{\theta}}=\cos\theta\Big|_{\theta_\circ}^{\theta}$ or $\dot{\theta}^2=2(\cos\theta-\cos\theta_\circ)$

$$\therefore T=4\int_0^{\theta_\circ}\frac{d\theta}{\left[2(\cos\theta-\cos\theta_\circ)\right]^{\frac{1}{2}}}$$

Time for pendulum to swing from $\theta=0$ to $\theta=\theta_\circ$ is $\frac{T}{4}$

Now—substitute $\sin\phi=\dfrac{\sin\frac{\theta}{2}}{\sin\frac{\theta_\circ}{2}}$ so $\phi=\frac{\pi}{2}$ at $\theta=\theta_\circ$

and use the identity $\cos\theta=1-2\sin^2\frac{\theta}{2}$ $\quad\therefore T=4\displaystyle\int_0^{\theta_\circ}\frac{d\theta}{\left[4\left(\sin^2\frac{\theta_\circ}{2}-\sin^2\frac{\theta}{2}\right)\right]^{\frac{1}{2}}}$

and after some algebra $\dfrac{d\phi}{\left[1-\sin^2\frac{\theta}{2}\right]^{\frac{1}{2}}}=\dfrac{d\theta}{\left[4\left(\sin^2\frac{\theta_\circ}{2}-\sin^2\frac{\theta}{2}\right)\right]^{\frac{1}{2}}}$ or

(a) $\quad T=4\displaystyle\int_0^{\frac{\pi}{2}}\frac{d\phi}{\left[1-\alpha\sin^2\phi\right]^{\frac{1}{2}}}$ where $\alpha=\sin^2\frac{\theta_\circ}{2}$

(b) $\quad\left(1-\alpha\sin^2\phi\right)^{-\frac{1}{2}}\approx 1+\frac{1}{2}\alpha\sin^2\phi+\frac{3}{8}\alpha^2\sin^4\phi+\ldots$

$T=4\displaystyle\int_0^{\frac{\pi}{2}}d\phi\left[1+\frac{1}{2}\alpha\sin^2\phi+\frac{3}{8}\alpha^2\sin^4\phi+\ldots\right]$ $\qquad T=2\pi\left[1+\frac{\alpha}{4}+\frac{9}{64}\alpha^2+\ldots\right]$

(c) $\quad\alpha=\sin^2\frac{\theta_\circ}{2}\approx\left[\frac{\theta_\circ}{2}-\frac{\theta_\circ^3}{48}+\ldots\right]^2\approx\frac{\theta_\circ^2}{4}\ \ldots$

$$T=2\pi\left[1+\frac{\theta_\circ^2}{16}+\ldots\right]$$

Continuous Systems (WAVES.)

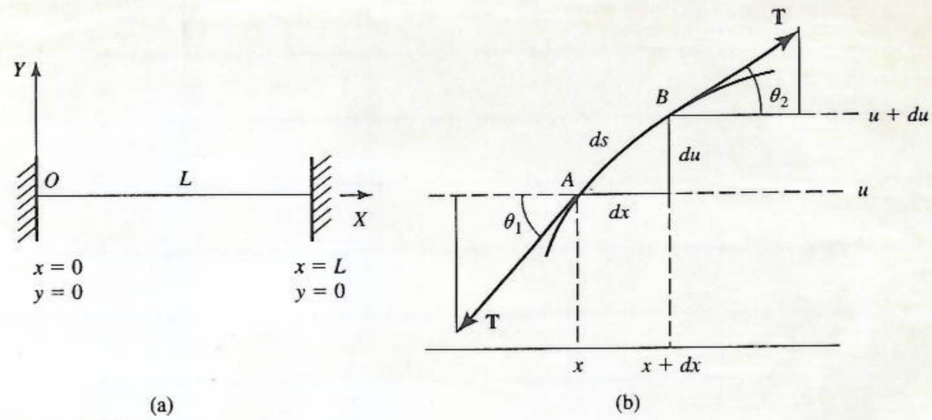


Figure 15.1 **(a)** A string of length L is horizontal when in equilibrium. **(b)** A small portion ds of a string under a small displacement results in transverse vibrations.

Consider a homogeneous string of length L that is fixed at both ends: $x = 0$ and $x = L$. The string has a linear density (mass per unit length) μ and is under tension T throughout the string. The string is in equilibrium along the x-axis as is shown in the figure above (a).

Now let us investigate the motion of such a string following an initial lateral displacement from its equilibrium position. We shall assume that the displacements of the string are not large enough to appreciably change the tension T. We shall further assume that the force due to gravity ($= \mu L g$) is small compared to the tension T and this may be neglected.

Now consider a small portion of the string AB of length ds having a horizontal length of dx between x and $x+dx$. For small displacements the tension remains the same.

$$\sum F_x = T\cos\theta_2 - T\cos\theta_1$$

$$\sum F_y = T\sin\theta_2 - T\sin\theta_1$$

If θ_1 and θ_2 are small, $\cos\theta_1 \simeq \cos\theta_2$, and hence there is no net horizontal force ($|\cos\theta_1| \simeq 1$) and therefore there is no longitudinal displacement of the string. That is to say, for small displacements of the string, we are concerned ONLY with transverse motion (motion perpendicular to the length of the string).

Because θ_1 and θ_2 are small

$$\sin\theta_1 \simeq \tan\theta_1 \quad \& \quad \sin\theta_2 \simeq \tan\theta_2$$

This implies

$$(1) \qquad \sum F_y \simeq T\tan\theta_2 - T\tan\theta_1$$

The motion of the string is described by a displacement function $u(x,t)$ of each point x and at each instant of time t!

Let the displacement of the string be u at x and $u+du$ at $x+dx$. According to Newton's 2nd Law

$$\text{(2)} \qquad \sum F_y = ma_y = m\ddot{u} = m\frac{\partial^2 u}{\partial t^2},$$

where m is the mass of a string of length AB, $m = \mu dx$, and $u = u(x,t)$ is the lateral displacement of the string at position x and instant of time t. Combining the preceding equations and setting $ds \simeq dx$ gives us

$$\text{(3)} \qquad \mu dx \frac{\partial^2 u}{\partial t^2} = T\tan\theta_2 - T\tan\theta_1 .$$

Using
$$T\tan\theta = T\frac{\partial u}{\partial x}$$
we may write the net vertical force as

$$\text{(4)} \qquad T\tan\theta_2 - T\tan\theta_1 = T\left(\frac{\partial u}{\partial x}\right)_B - T\left(\frac{\partial u}{\partial x}\right)_A .$$

The slope of the string at B may be expanded by means of a Taylor series:

$$\text{(5)} \qquad \left(\frac{\partial u}{\partial x}\right)_B = \left(\frac{\partial u}{\partial x}\right)_A + \left(\frac{\partial^2 u}{\partial x^2}\right)_A dx + \cdots$$

Inserting (5) into (4) and equating to (3) gives

$$\mu\, dx \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2} dx$$

or

$$\frac{\partial^2 u}{\partial x^2} = \frac{\mu}{T} \frac{\partial^2 u}{\partial t^2}$$

Since the dimensions of μ are $[ML^{-1}]$ and the dimensions of force are $[MLT^{-2}]$, the dimensions of μ/T are $[L^{-2}T^{2}]$ or the reciprocal of velocity squared. Hence the **wave equation** of the vibrating string is

(*) $$\boxed{\frac{\partial^2 u}{\partial x^2} - \frac{1}{V^2}\frac{\partial^2 u}{\partial t^2} = 0}$$ where $V = \sqrt{T/\mu}$

V is not simply a velocity of propagation; it holds a much deeper physical interpretation, which we shall explore later. For now we shall call it the **wave velocity**, that is the velocity with which the disturbance propagates along the string.

General Solution: Normal Modes of Vibration

The wave equation (*) is a partial differential equation for the function $u(x,t)$ that describes the motion of the vibrating string. To evaluate the function $u(x,t)$, we make use of initial and boundary conditions.

Suppose at $t=0$ the function $u(x,t)$ satisfies the following initial conditions

$$u(x,0) = u_0(x) \leftarrow \text{Displacement of the string at } t=0$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \dot{u}_0(x) \leftarrow \text{velocity of the string at } t=0$$

N.B. Both are functions of the position x.

Since the string is secured at both ends, it must satisfy the boundary conditions:

$$u(0,t) = u(L,t) = 0$$

That is to say, the displacement at the ends IS ZERO AT ALL TIMES.

We shall now proceed to find the solution $u(x,t)$ of the differential equation (*). We shall make use of the

METHOD of SEPARATION of VARIABLES

Let $u(x,t) = X(x)\,\Theta(t)$ where X is a function of x alone and Θ is a function of t alone. We have then

$$\frac{\partial^2 u}{\partial x^2} = \Theta \frac{d^2 X}{dx^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} = X \frac{d^2 \Theta}{dt^2} \qquad (6)$$

Substituting (6) into the wave equation (*) and rearranging gives

$$\frac{V^2}{X}\frac{d^2 X}{dx^2} = \frac{1}{\Theta}\frac{d^2 \Theta}{dt^2} \quad \longleftarrow \; !$$

The left side of this equation is a function of x only, while the right side is a function of t only! This is possible for all values of x and t if AND ONLY if each side is equal to a constant. Let this constant be $-\omega^2$. The minus sign indicates that the acceleration of the element of the string is ALWAYS directed towards the equilibrium position. That is, the acceleration is opposite to the displacement.

Therefore from (7) we have

$$\frac{V^2}{X}\frac{d^2X}{dx^2} = -\omega^2 \quad \text{or} \quad \frac{d^2X}{dx^2} + \frac{\omega^2}{V^2}X = 0 \tag{8}$$

and

$$\frac{1}{\Theta}\frac{d^2\Theta}{dt^2} = -\omega^2 \quad \text{or} \quad \frac{d^2\Theta}{dt^2} + \omega^2\Theta = 0 \tag{9}$$

where ω may be interpreted as the angular frequency.

The solution to (8) is

$$X(x) = C\cos\frac{\omega}{V}x + D\sin\frac{\omega}{V}x \tag{10}$$

and the solution for (9) is

$$\Theta(t) = E\cos\omega t + F\sin\omega t \tag{11}$$

Here C, D, E, and F are the four constants of integration which are to be evaluated by using the initial and boundary conditions.

Since $u(x,t) = X(x)\,\Theta(t)$, we have

(12) $$u(x,t) = \left(C\cos\frac{\omega}{V}x + D\sin\frac{\omega}{V}x\right)\left(E\cos\omega t + F\sin\omega t\right)$$

We may now apply the boundary conditions to evaluate the constants C and D. At $x=0$, $u(0,t)=0$ for all values of t; that is $X(0)=0$ or

$$0 = C\cos\left(\frac{\omega}{v}0\right) + D\sin\left(\frac{\omega}{v}0\right)$$

which is possible if $C=0$. Therefore

$$X(x) = D\sin\frac{\omega}{v}x$$

At $x=L$, $u(L,t)=0$ $\forall$ values of t, i.e. $X(L)=0$. This implies

$$0 = D\sin\frac{\omega}{v}L$$

Neglecting the trivial solution of $D=0$, we must have

$$\sin\frac{\omega}{v}L = 0 \quad \text{or} \quad \frac{\omega}{v}L = n\pi \quad (n=1,2,3,\ldots)$$

Replacing ω by ω_n, and $v=\sqrt{T/\mu}$ gives:

$$\omega_n = \frac{n\pi v}{L} = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}} \qquad (13)$$

Thus, with $C=0$ and letting $DE=A_n$ and $DF=B_n$, we may write (from (12))

(14) $$u(x,t) = \left(A_n\cos\omega_n t + B_n\sin\omega_n t\right)\sin\frac{n\pi}{L}x$$

or since $\omega_n = \frac{n\pi}{L} v$

(14') $$u(x,t) = A_n \sin \frac{n\pi x}{L} \cos \frac{n\pi v}{L} t + B_n \sin \frac{n\pi x}{L} \sin \frac{n\pi v}{L} t$$

Equations (14) & (14') represent the NORMAL MODE OF VIBRATION of the string, in particular, the n^{th} mode. The velocity of the normal mode can be obtained by differentiating (14')

(15) $$\dot{u}(x,t) = A_n \left(-\frac{n\pi v}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi v}{L} t\right) + B_n \left(\frac{n\pi v}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi v}{L} t\right)$$

We can now evaluate the constants A_n and B_n of the n^{th} mode of vibration by making use of the initial conditions at $t=0$

$$u(x,0) = u_0(x) \quad \text{and} \quad \dot{u}(x,0) = \dot{u}_0$$

From these conditions in equations (14') and (15) we obtain

$$u_0(x) = A_n \sin \frac{n\pi x}{L}$$

$$\dot{u}_0(x) = \frac{n\pi v}{L} B_n \sin \frac{n\pi x}{L}$$

We know from the theory of differential equations that if $u_1(x,t)$ and $u_2(x,t)$ are any two solutions which satisfy the boundary conditions, then the linear combination of these two solutions is also a solution $u(x,t) = u_1(x,t) + u_2(x,t)$.

A more general solution is obtained by adding together all the n particular solutions corresponding to the different frequencies ω_n. THUS the general solution of motion of a vibrating string is a linear combination of a large number of normal modes (cf. eqn 14) and is given by

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \sin\frac{n\pi x}{L}\cos\omega_n t + B_n \sin\frac{n\pi x}{L}\sin\omega_n t\right) \tag{16}$$

where $\omega_n = \dfrac{n\pi v}{L}$, $n = 1, 2, 3, \ldots$

Note that eqn (16) contains an infinite number of arbitrary constants. Again the initial conditions are (at $t=0$)

$$u(x,0) = u_0(x)$$

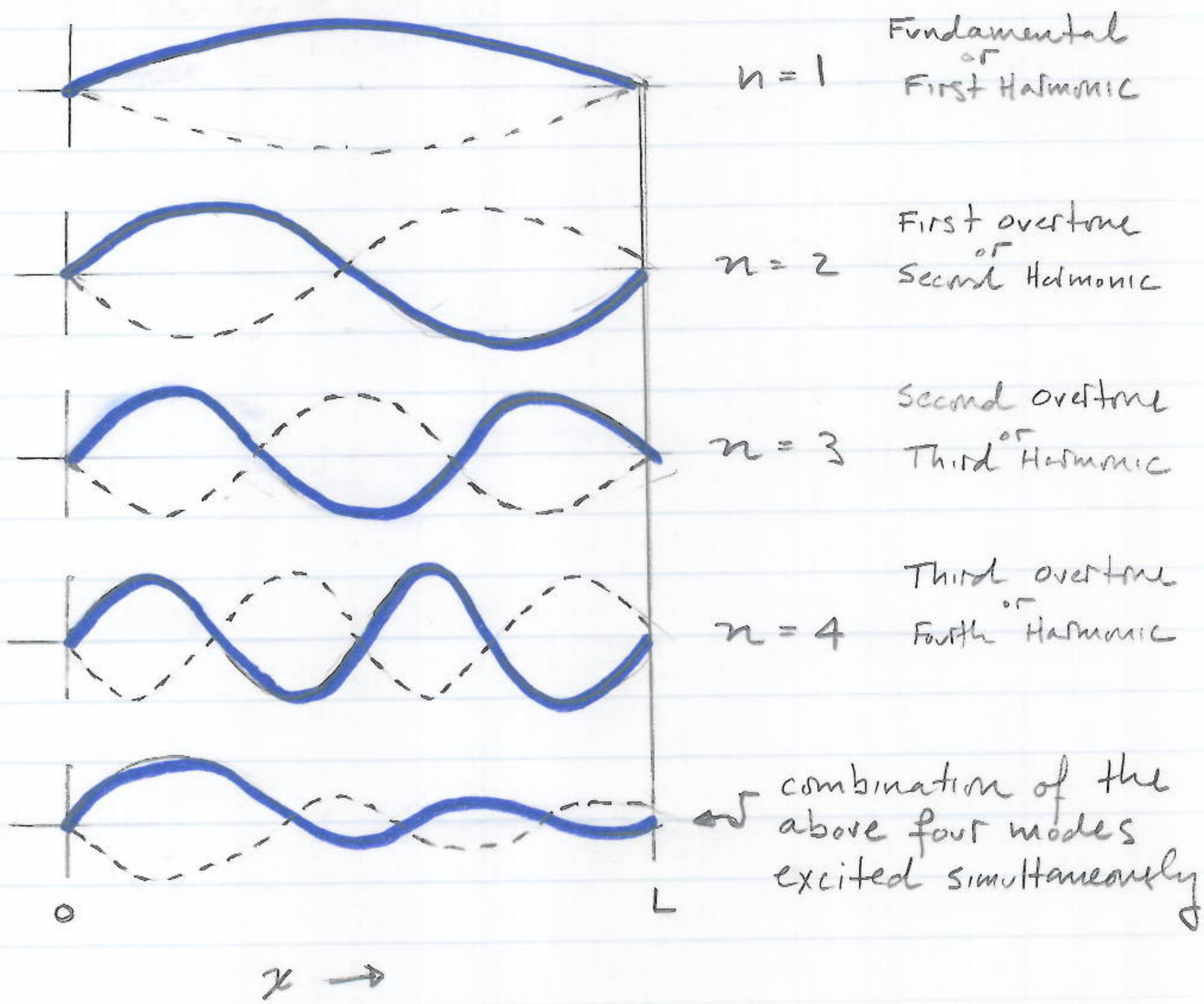
$$\dot{u}(x,0) = \dot{u}_0(x)$$

Then from eqn (16) we have

$$u_0(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \qquad (17a)$$

$$\dot{u}_0(x) = \sum_{n=1}^{\infty} \frac{n\pi v}{L} B_n \sin \frac{n\pi x}{L} \qquad (17b)$$

Before we get involved in evaluating the constants, let's look at the first few harmonics



"Standing Waves"

In general, a string vibrates with several modes simultaneously.

The general solution given by equation (16) is a FOURIER SERIES which consists of a sum of sines and/or cosines. The general solution is completely known if the coefficients A_n and B_n are known and these coefficients can be evaluated if the initial conditions (i.e. $u_0(x)$ and $\dot{u}_0(x)$) are known. We shall use the FOURIER TECHNIQUE to evaluate these constants A_n & B_n.

Multiplying both sides of equ. (17a) by $\sin\left(\frac{m\pi x}{L}\right)$, where m is an integer, and integrating from $x=0$ to $x=L$

$$\int_0^L u_0(x) \sin\frac{m\pi x}{L}\,dx$$

$$= \int_0^L \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

But all terms on the right will vanish unless $m=n$

$$\int_0^L u_0(x)\sin\frac{n\pi x}{L}\,dx = A_n\int_0^L \sin^2\frac{n\pi x}{L}\,dx$$

$$= A_n\frac{L}{2}$$

OR

(18a) $$A_n = \frac{2}{L}\int_0^L u_0(x) \sin \frac{n\pi x}{L}\, dx$$

Similarly, multiplying both sides of equ. (17b) by $\sin\left(\frac{m\pi x}{L}\right)$ and integrating from $x=0$ to $x=L$

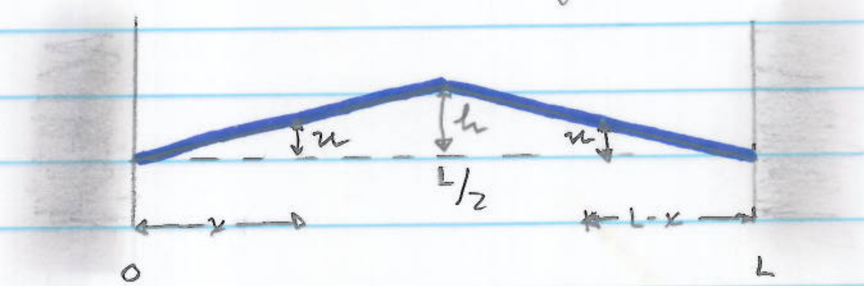
$$\int_0^L \dot{u}_0(x) \sin \frac{m\pi x}{L}\, dx = \int_0^L \sum_{n=1}^{\infty} \frac{n\pi v}{L} B_n \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx$$

yields

(18b) $$B_n = \frac{2}{n\pi v}\int_0^L \dot{u}_0(x) \sin \frac{n\pi x}{L}\, dx.$$

Thus equations 18a and 18b state that if the displacement $u_0(x)$ and the velocity $\dot{u}_0(x)$ are given for all points of the string at one time (a snapshot in time, if you will), A_n and B_n can be evaluated. Once these constants are known, the motion of the string is determined for all subsequent time.

Example A string of length L and having a mass per unit length of μ is fixed at both ends and is under tension T. The string is initially displaced a distance h ($h \ll L$) at the middle of the string and is then released. Evaluate the Fourier coefficients for the subsequent motion of the string.



The initial conditions are (cf Fig. above)

- For $0 < x < \frac{L}{2}$ $\qquad \frac{u}{x} = \frac{h}{L/2} \Rightarrow u = \frac{2h}{L}x \qquad (1)$

- For $\frac{L}{2} < x < L$ $\qquad \frac{u}{L-x} = \frac{h}{L/2} \Rightarrow u = \frac{2h}{L}(L-x) \qquad (2)$

At $t=0$ $\quad \frac{du_0}{dt} = \dot{u}_0 = 0 \qquad (3)$

$$\dot{u}_0 = 0 \Rightarrow B_n = 0$$

and

$$A_n = \frac{2}{L}\int_0^L u_0 \sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\left[\frac{2h}{L}\int_0^{L/2} x \sin\frac{n\pi x}{L}\,dx + \frac{2h}{L}\int_{L/2}^{L}(L-x)\sin\frac{n\pi x}{L}\,dx\right]$$

Evaluating the integrals for different values of n, we obtain

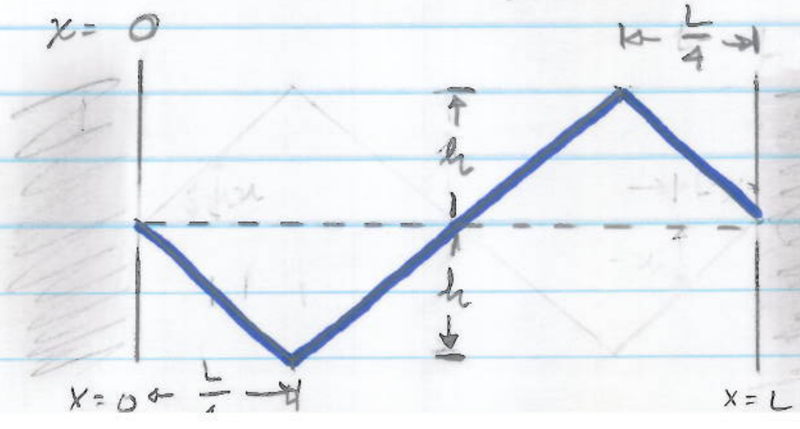
$$A_n = \begin{cases} 0, & n \text{ even} \\ \dfrac{8h}{n^2\pi^2} \sin \dfrac{n\pi}{2} & n \text{ odd} \end{cases}$$

Substituting our values of A_n & B_n into equation (18) on p. 13-10, we obtain

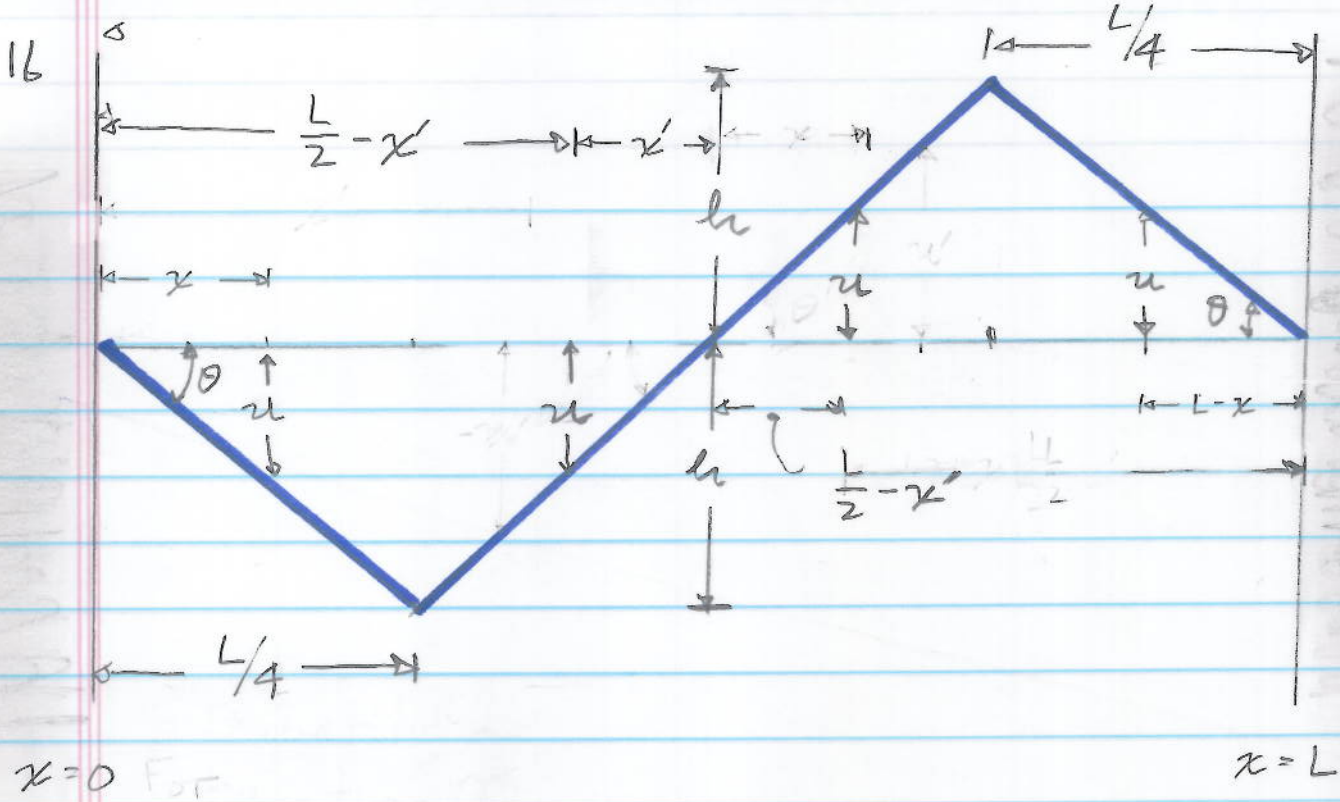
$$u(x,t) = \frac{8h}{\pi^2}\left\{ \sum_{n\ odd} \frac{(-1)^{\frac{n-1}{2}}}{n^2} \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{n\pi v}{x}t\right) \right\}$$

Note that only the odd harmonics have been excited.

<u>Example</u> A string is plucked a distance h at a point $L/4$ from one end. At a point $L/4$ from the other end, the string is pulled aside a distance h in the opposite direction. Discuss the vibrations in terms of the normal modes.



13-16



$\boxed{\text{Case 1}}$ $\quad 0 < x < \frac{L}{4}$

$$\frac{u}{x} = -\frac{h}{L/4} \Rightarrow u = -\frac{4h}{L}x$$

$\boxed{\text{Case 2}}$ $\quad \frac{3L}{4} < x < L$

$$\frac{u}{L-x} = \frac{h}{L/4} \Rightarrow u = \frac{4h}{L}(L-x)$$

$\boxed{\text{CASE 3}}$ $\quad \frac{L}{4} < x < \frac{L}{2}$

$$\frac{u}{\left(\frac{L}{2}-x\right)} = -\frac{h}{L/4} \Rightarrow u = -\frac{4h}{L}\left(\frac{L}{2}-x\right)$$

$$= \frac{2h}{L}(2x-L)$$

$\boxed{\text{CASE 4}}$ $\quad \frac{L}{2} < x < \frac{3L}{4}$

$$\frac{u}{\left(x-\frac{L}{2}\right)} = \frac{h}{L/4} \qquad u = \frac{4h}{L}\left(x-\frac{L}{2}\right)$$

$$u = \frac{2h}{L}(2x-L)$$

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$$u = \begin{cases} -\frac{4h}{L}x & 0 < x < \frac{L}{4} \quad (i) \\ \frac{2h}{L}(2x - L) & L/4 < x < 3L/4 \quad (ii) \\ \frac{4h}{L}(L - x) & 3L/4 < x < L \quad (iii) \end{cases}$$

check:

$x = 0 \Rightarrow u = 0$ from (i)

$x = L/4 \Rightarrow u = -h$ from (i)

$\Rightarrow u = -h$ from (ii)

$x = \frac{3L}{4} \Rightarrow u = +h$ from (ii)

$\Rightarrow u = +h$ from (iii)

✓

At $t = 0$ $\quad \dot{u}_0 = 0 \Rightarrow B_n = 0$

And

$$A_n = \frac{2}{L}\left\{\int_0^L u_0 \sin\left(\frac{n\pi x}{L}\right)dx\right\}$$

$$= \frac{2}{L}\left\{-\frac{4h}{L}\int_0^{L/4} x\sin\left(\frac{n\pi x}{L}\right)dx + \frac{2h}{L}\int_{L/4}^{3L/4}(2x-L)\sin\left(\frac{n\pi x}{L}\right)dx + \frac{4h}{L}\int_{3L/4}^{L}(L-x)\sin\left(\frac{n\pi x}{L}\right)dx\right\}$$

$$= \frac{8h}{L^2}\left\{\underset{①}{-\int_0^{L/4} x\sin\left(\frac{n\pi x}{L}\right)dx} + \underset{②}{\int_{L/4}^{3L/4} x\sin\left(\frac{n\pi x}{L}\right)dx} \underset{③}{-\frac{L}{2}\int_{L/4}^{3L/4}\sin\left(\frac{n\pi x}{L}\right)dx} + \underset{④}{L\int_{3L/4}^{L}\sin\left(\frac{n\pi x}{L}\right)dx} \underset{⑤}{-\int_{3L/4}^{L} x\sin\left(\frac{n\pi x}{L}\right)dx}\right\}$$

We observe that

$$\int x \sin(ax)\,dx = \frac{1}{a^2}\sin ax - \frac{x}{a}\cos ax$$

and $$\int \sin(ax)\,dx = -\frac{1}{a}\cos ax$$

For our integrals let $a = \frac{n\pi}{L}$

$$-\int_{①} = -\frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{4}\right) + \frac{L^2}{4n\pi}\cos\left(\frac{n\pi}{4}\right)$$

$$\int_{②} = \frac{L^2}{n^2\pi^2}\sin\left(\frac{3n\pi}{4}\right) - \frac{3L^2}{4n\pi}\cos\left(\frac{3n\pi}{4}\right)$$
$$- \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{4}\right) + \frac{L^2}{4n\pi}\cos\left(\frac{n\pi}{4}\right)$$

$$-\frac{L}{2}\int_{③} = \frac{L^2}{2n\pi}\cos\left(\frac{3n\pi}{4}\right) - \frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{4}\right)$$

$$L\int_{④} = \frac{L^2}{n\pi}\cos\left(\frac{3n\pi}{4}\right) - \frac{L^2}{n\pi}\cos(n\pi)$$

$$-\int_{⑤} = \frac{L^2}{n\pi}\cos(n\pi) + \frac{L^2}{n^2\pi^2}\sin\left(\frac{3n\pi}{4}\right) - \frac{3L^2}{4n\pi}\cos\left(\frac{3n\pi}{4}\right)$$

$$A_n = \frac{8h}{L^2}\left\{-\int_{①} + \int_{②} - \frac{L}{2}\int_{③} + L\int_{④} - \int_{⑤}\right\}$$

$$A_n = \frac{16h}{n^2\pi^2}\left[\sin\left(\frac{3n\pi}{4}\right) - \sin\left(\frac{n\pi}{4}\right)\right]$$

Clearly $A_n = 0$ when $\frac{3n}{4}$ and $\frac{n}{4}$ are simultaneously equal to integers. This will occur when n is a multiple of 4. We may therefore conclude that the modes with frequencies that are multiples of $4\omega_1$ will be absent.

Damping & Driving Terms

Now let us imagine there is damping present and that the string is being driven by an external force, $F_e(x,t)$

Let the damping force be proportional to the velocity

$$F_d = -D\dot{u} \quad \text{(resistive term)}$$

From Newton's Second Law:

$$\sum F = m\frac{\partial^2 u}{\partial t^2}$$

From our results on p. 13-4 and including the external driving force and the resistive term, we find the relationship.

$$T\frac{\partial^2 u}{\partial x^2}dx - D\frac{\partial u}{\partial t}dx + F_e\,dx = \mu\,dx\,\frac{\partial^2 u}{\partial t^2}$$

or

(19) $$\mu\frac{\partial^2 u}{\partial t^2} + D\frac{\partial u}{\partial t} - T\frac{\partial^2 u}{\partial x^2} = F_e$$ Here $F_e = F_e(x,t)$

Let us consider the case wherein the string IS NOT DRIVEN by an external force; $F_e = 0$

Let us assume a solution

(20) $$u(x,t) = \sum_n \eta_n(t)\,\sin\left(\frac{n\pi}{L}x\right)$$

Here $\eta_n = \eta_n(t)$; a function of time ONLY.

For example, in the case of no damping present, we found by means of the technique of separation of variables that (cf. equ. 14 on p. 13-18)

$$\eta_n = A_n\cos\omega_n t + B_n\sin\omega_n t$$

Inserting (20) into (19) and setting $F_e = 0$ gives

(21) $$\sum_{n=1}^{\infty}\left[\left(\mu\ddot{\eta}_n + D\dot{\eta}_n + \frac{n^2\pi^2 T}{L^2}\eta_n\right)\sin\left(\frac{n\pi}{L}x\right)\right] = 0$$

Multiplying equ. (21) by $\sin(\frac{m\pi}{L}x)$ and integrating over the length of the string:

$$0 = \sum_{n=1}^{\infty} \int_0^L \left[\left[\mu\ddot{\eta}_n + D\dot{\eta}_n + \frac{n^2\pi^2 T}{L^2}\eta_n \right] \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) \right] dx$$

But $$\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right) dx = \frac{L}{2}\delta_{mn}$$

$$\delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

Therefore

$$\ddot{\eta}_m + \frac{D}{\mu}\dot{\eta}_m + \frac{m^2\pi^2 T}{\mu L^2}\eta_m = 0$$

Let $\frac{D}{\mu} = 2\gamma$ and $\omega_0^2 = \frac{m^2\pi^2 T}{\mu L^2}$

we found in Ch. 3 that the solution to this differential equation is

$$\eta(t) = e^{-\gamma t}\left[A_1 e^{\lambda t} + A_2 e^{-\lambda t} \right]$$

Here $\lambda^2 = \gamma^2 - \omega_0^2$

Hence

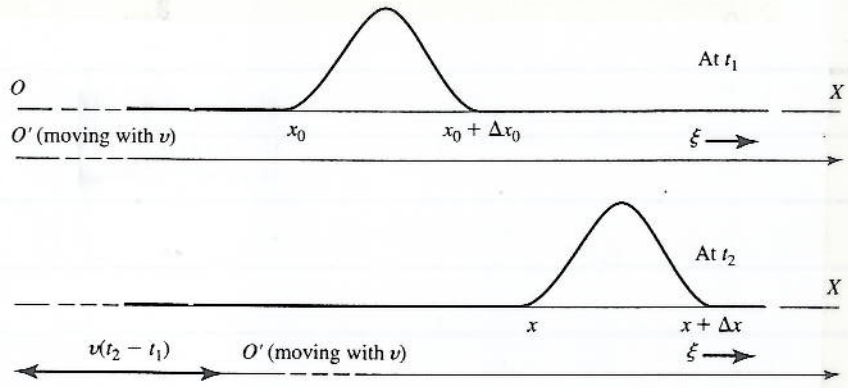
$$\eta_m(t) = e^{-\frac{D}{2\mu}t}\left\{A_1 \exp\left[\sqrt{\frac{D^2}{4\mu^2} - \frac{m^2\pi^2 T}{\mu L^2}}\, t\right] + A_2 \exp\left[-\sqrt{\frac{D^2}{4\mu^2} - \frac{m^2\pi^2 T}{\mu L^2}}\, t\right]\right\}$$

WAVE PROPAGATION IN GENERAL

A more refined definition of wave motion is from the viewpoint of ENERGY TRANSPORT. When a wave reaches a portion of a medium, it sets the particles in that medium into motion. After the wave has passed those particles come to rest, while the neighboring particles are set into motion.

$$\boxed{\text{WAVE MOTION PROVIDES A MECHANISM FOR THE TRANSFER OF ENERGY FROM ONE POINT TO ANOTHER \underline{WITHOUT} THE PHYSICAL TRANSFER OF \underline{ANY} \underline{MATERIAL} BETWEEN THE POINTS.}}$$

13-23



Pulse in a rope traveling to the right and viewed by an observer moving with velocity v along an axis parallel to the rope.

Let us discuss the propagation of a single pulse in one dimension. Consider a stretched rope that has been shaken at one end, resulting in a pulse traveling along its length and taking the form as depicted in the figure above. This PULSE (WAVE or DISTURBANCE) travels along the rope without distortion in form. That is to say it maintains the same shape for all time. (This is of course an idealized case, for in practice there will be damping which will tend to distort this waveform). This pulse will travel with constant velocity along the x-axis.

WAVE MOTION is a disturbance that propagates through the medium with constant velocity without any change in its form or pattern

Let us suppose that this PULSE or DISTURBANCE is traveling along the x-axis with constant velocity v. We now view this pulse from the ξ-axis, which is moving with velocity v along and parallel to the x-axis. If the origins of the x- and ξ-axes coincide at $t=0$, we have

$$\xi = x - vt$$

To an observer in the ξ system, the form and position of the disturbance remains unchanged. That is, the disturbance has a time dependence which is a function of ξ alone.

$$u(x,t) = f(\xi) \equiv f(x-vt) \qquad (22)$$

where $f(\xi)$ represents the wave traveling to the right. Equation (22) garentees that it is a wave traveling to the right, for as t increases so must x so that ξ remains constant. Similarly we may define

$$\eta = x + vt$$

and a wave traveling to the left is given by

$$u(x,t) = g(\eta) \equiv g(x+vt) \qquad (23)$$

Here, $g(\eta)$ represents a wave propagating to the left. As t increasings, x must decrease so that η is a constant.

f and g given by equations (22) and (23) are referred to as **WAVE FORMS** and represent the most general type of one-dimensional motion.

The general expression for u is a combination of two functions: one which depends upon ξ and the other one which depends upon η. That is, the sum of the two linear functions of ξ and η.

$$u(x,t) = f(\xi) + g(\eta)$$

$$= f(x - vt) + g(x + vt)$$

This is consistent with the fact that the general solution of a second-order differential equation is composed of two functions.

Let us now evaluate these functions, g and f by making use of the initial conditions:

$$u = u_0(x) \quad \text{and} \quad \dot{u} = \dot{u}_0(x)$$

at $t = 0$.

This gives

(24) $$u(x,0) = f(x) + g(x) = u_0(x)$$

and

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \frac{\partial}{\partial t}(f+g) = \frac{df}{d\xi}\frac{\partial \xi}{\partial t} + \frac{dg}{d\eta}\frac{\partial \eta}{\partial t}$$

$$= \left[-v\frac{df}{d\xi} + v\frac{dg}{d\eta}\right]_{t=0} = \dot{u}_0(x)$$

At $t=0$, $\xi = \eta = x$, and we have

$$v\frac{d}{dx}\left[-f(x) + g(x)\right] = \dot{u}_0(x)$$

which upon integration gives

(25) $$-f(x) + g(x) = \frac{1}{v}\int_0^x \dot{u}_0(x)\,dx + C$$

constant of integration.

Adding and subtracting equations (24) & (25)

$$f(x) = \frac{1}{2}\left[u_0(x) - \frac{1}{v}\int_0^x \dot{u}_0(x)\,dx - C\right]$$

$$g(x) = \frac{1}{2}\left[u_0(x) + \frac{1}{v}\int_0^x \dot{u}_0(x)\,dx + C\right]$$

Since these solutions hold for any value of x, we may replace x by ξ or η. Also the constant C may be dropped because it will be eliminated in the linear combinations of the solutions. Thus

$$f(\xi) = \frac{1}{2}\left[u_0(\xi) - \frac{1}{V}\int_0^{\xi} \dot{u}_0(\xi)\,d\xi\right]$$

$$g(\eta) = \frac{1}{2}\left[u_0(\eta) + \frac{1}{V}\int_0^{\eta} \dot{u}_0(\eta)\,d\eta\right]$$

Our next step is to make the connection between the general solution just derived and those solutions for the vibrating string. We found (p. 13-7)

$$\frac{d^2 X}{dt^2} + \frac{\omega^2}{V^2} X = 0 \qquad (8)$$

$$\frac{d^2 \Theta}{dt^2} + \omega^2 \Theta = 0 \qquad (9)$$

Instead of writing the solutions in terms of sines and cosines, let us write them as:

$$X(x) = C e^{i\frac{\omega}{V}x} + D e^{-i\frac{\omega}{V}x}$$

$$\Theta(t) = E e^{i\omega t} + F e^{-i\omega t}$$

Here C, D, E, and F are constants to be determined from the initial conditions. The general solution will be of the form

$$u(x,t) = X(x)\,\Theta(t) = A e^{\pm i \frac{\omega}{v} x} e^{\pm i\omega t}$$

$$= A e^{\pm i \frac{\omega}{v}(x \pm vt)} \qquad [A \text{ is a constant}]$$

In this notation, the wave function $u(x,t)$ is a linear combination of

$$\exp\left[i \frac{\omega}{v}(x+vt)\right]$$
$$\exp\left[i \frac{\omega}{v}(x-vt)\right]$$
$$\exp\left[-i \frac{\omega}{v}(x+vt)\right]$$
$$\exp\left[-i \frac{\omega}{v}(x-vt)\right]$$

The solutions containing $x-vt$ represent waves traveling to the right. And those containing $x+vt$ represent waves traveling to the left.

THESE SOLUTIONS REPRESENT TRAVELING WAVES.

Furthermore, these equations are not satisfied by one particular value of ω_n. Because we have not specified any boundary conditions, there exist an infinite number of frequencies.

The general solution is not only a linear combination of harmonic terms, but also must be summed over all possible frequencies

$$u(x,t) = \sum_n A_n e^{\pm i \frac{\omega_n}{v}(x \pm vt)}$$

Once the boundary conditions are specified, the constants can be evaluated in a manner similar to the case of determining the coefficients in an infinite Fourier series.

For the following discussion we shall write the solution as

$$u(x,t) = A e^{i \frac{\omega}{v}(x - vt)} \qquad (26)$$

with it being explicitly understood that the complete solution is to be summed over all frequencies.

The quantity k is called the **propagation constant** or **wave number** (number of waves per unit length). It has dimensions of reciprocal length and is defined:

$$k^2 \equiv \frac{\omega^2}{v^2} \quad \text{or} \quad |k| = \frac{\omega}{v} \qquad (27)$$

The wave equation for X (equ. (8)) and its general solution (equ (26)) take the forms

$$\frac{d^2 X}{dx^2} + k^2 X = 0 \qquad \text{[Helmholz Equation]}$$

and

$$u(x,t) = Ae^{ik(x-vt)} = Ae^{i(kx-\omega t)}$$

If ν is the frequency of vibration so that $\omega = 2\pi\nu$, then the wavelength λ is defined as the distance for one complete vibration of the wave

$$\lambda = \frac{v}{\nu} = \frac{2\pi v}{2\pi\nu} = \frac{2\pi v}{\omega}$$

Combining this with our definition for k (equ. (27)), we see

$$\lambda = \frac{2\pi}{k} \text{ or } k = \frac{2\pi}{\lambda}$$

Now let us see what happens when we superimpose two waves, each having the same frequency and amplitude, but with one traveling to the right and the other to the left.

Mathematically:

$$u = u_1 + u_2 = Ae^{i(kx-\omega t)} + Ae^{i(kx+\omega t)}$$

$$= Ae^{ikx}(e^{-i\omega t} + e^{+i\omega t}) = 2Ae^{ikx}\cos\omega t$$

$$= 2A\cos\omega t\left[\cos kx + i\sin kx\right]$$

The real part of this equation is

$$\mathrm{Re}\{u\} = 2A\cos kx\cos\omega t. \quad (28)$$

This wave has the property that it does not propagate forward with time. This superposition of these two waves leads to the formation of **STANDING WAVES**. There are certain points where there is no motion at all due to the complete cancellation of one wave by the other. Such points are called **NODES**. From equ (28) we can obtain where these nodes are located

$$kx = (n + \tfrac{1}{2})\pi$$

$$x = (2n+1)\frac{\pi}{2k} = (2n+1)\frac{\lambda}{4}$$

Here n is an integer.

PHASE VELOCITY & DISPERSION

Let us consider a wave of single frequency given by

$$u(x,t) = Ae^{i(kx-\omega t)}$$

If the argument of this exponential remains constant, the wave function $u(x,t)$ also remains constant. The quantity $kx-\omega t$ is defined as the **PHASE** of the wave represented by $u(x,t)$.

$$\phi \equiv kx - \omega t$$

If we move along the x-axis at such a velocity so that the phase at every point is the same, the wave pattern or form will remain unchanged with time. For ϕ to remain constant, we must have

$$\Delta\phi = d\phi = 0 \Rightarrow k\,dx - \omega\,dt = 0$$

From which we define the phase velocity V_p to be the velocity with which the wave pattern travels

$$V_p = \frac{dx}{dt} = \frac{\omega}{k} = V$$

That is, for a simple wave possessing a well-defined frequency, the phase velocity v_p is equal to the wave velocity v. In general, this is not true.

Usually the phase velocity is a function of frequency

$$v_p = v_p(k)$$

Such a medium, wherein $v_p = v_p(k)$, is called **DISPERSIVE**. In a dispersive medium the phase velocity is not equal to the wave velocity. ¶(The best-known example of this phenomenon is the simple optical prism. The index of refraction depends upon the wavelength of the incident light).

In dispersive media the wave pattern is modified; it does not remain constant.

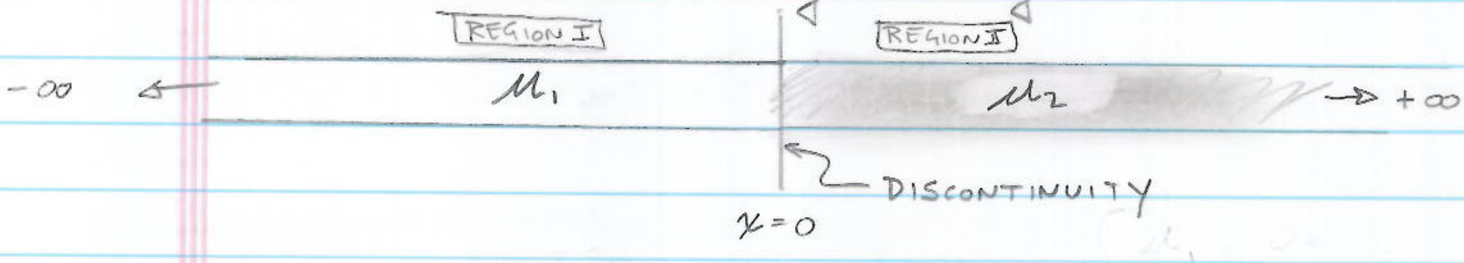
But even such a pattern will appear unchanged to an observer who is moving with a velocity v_g given by

$$v_g = \frac{d\omega(k)}{dk} \qquad \text{Here } \omega = \omega(k)$$

GROUP VELOCITY

Wave at a discontinuity: Energy Flow

Consider two semi-infinite strings of different linear mass densities joined together at $x=0$



$$\mu = \begin{cases} \mu_1 & -\infty < x < 0 \\ \mu_2 & 0 < x < \infty \end{cases} \quad \leftarrow \text{linear mass density}$$

$$V = \begin{cases} V_1 & -\infty < x < 0 \\ V_2 & 0 < x < \infty \end{cases} \quad \leftarrow \text{wave velocity}$$

The incoming wave traveling from the left will be both reflected and transmitted at $x=0$, where the mass discontinuity occurs.

<u>Goal</u>: Calculate the reflected and transmitted amplitudes.

<u>Region I</u> $\quad u_1(x,t) = u_{inc} + u_{refl}$

$$u_1(x,t) = Ae^{i(\omega t - k_1 x)} + Be^{i(\omega t + k_1 x)}$$

<u>Region II</u> $\quad u_2(x,t) = u_{trans} = Ce^{i(\omega t - k_2 x)}$

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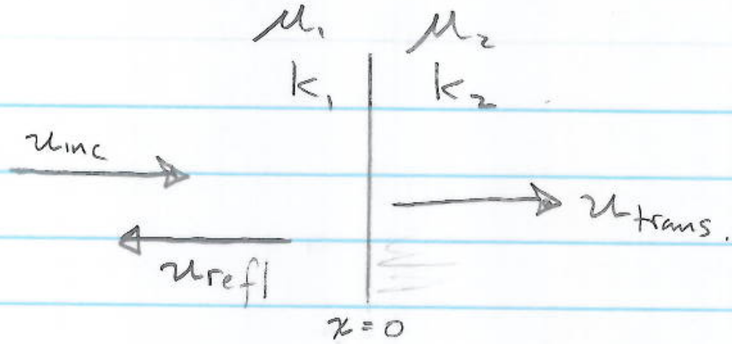

Boundary Conditions (CONTINUITY CONDITIONS)

① $\boxed{u_1 = u_2}$ (continuous across interface). This condition insures that there is no break in the string at the interface.

② $\boxed{\dfrac{\partial u_1}{\partial x} = \dfrac{\partial u_2}{\partial x}}$ (First derivative is continuous across the interface). This condition prevents a "kink" from occuring in the string @ $x=0$. This is to say the limit:

$$\lim_{\Delta x \to 0^+} \frac{\Delta u_1}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{\Delta u_2}{\Delta x}$$

If this derivative were not continuous, the second derivative of the wavefunction wrt x would be infinite and a finite force acting on the string element at this junction would produce an infinite acceleration.

Imposing the first continuity condition yields

$$A + B = C$$

And from the second continuity condition:

$$-k_1 A + k_1 B = -k_2 C$$

The solution of these pair of equations gives

$$\frac{B}{A} = \frac{k_1 - k_2}{k_1 + k_2}, \quad \frac{C}{A} = \frac{2k_1}{k_1 + k_2}$$

Since $k = \frac{\omega}{v}$ and $v = \sqrt{T/\mu}$ we have

$$\frac{B}{A} = \frac{v_2 - v_1}{v_1 + v_2} = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}}$$

$$\frac{C}{A} = \frac{2v_2}{v_1 + v_2} = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1} + \sqrt{\mu_2}}.$$

It is clear that the ratio $\frac{C}{A}$ is always positive. The transmitted wave is therefore always in phase with the incident wave. If the second medium is less dense ($\mu_2 < \mu_1$), the ratio B/A will be positive and the reflected wave will be in phase with the incident wave. If, on the other hand, the second medium is denser than the first (i.e. $\mu_2 > \mu_1$), the ratio B/A will be negative and the reflected wave will be π out of phase with respect to the incident wave. [Can you make analogies to electromagnetic waves traveling from one medium having an index of refraction n_1 to another of n_2?]

The intensity (the energy flow per unit time per unit area) for any type of wave motion is proportional to the square of the amplitude.

We define the **REFLECTION COEFFICIENT**, $\mathbb{R}$, to be the fraction of the incident energy that is reflected back

$$\mathbb{R} \equiv \left(\frac{B}{A}\right)^2 = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 = \left(\frac{V_2 - V_1}{V_1 + V_2}\right)^2$$

No energy may be stored in the junction of the two strings, so the sum of the reflected and transmitted energies must equal the incident energy ($\mathbb{R} + \mathbb{T} = 1$)

Thus

$$\mathbb{T} = 1 - \mathbb{R} = \frac{4k_1 k_2}{(k_1+k_2)^2} = \frac{4v_1 v_2}{(v_1+v_2)^2}$$

We observe that $\mathbb{R}$ becomes larger as the difference between μ_1 and μ_2 (or v_1 and v_2) becomes larger, while the corresponding $\mathbb{T}$ becomes smaller.

Now let us calculate the rate of energy flow across the junction at $x = 0$. This equals the time rate of change of the work done by the adjacent element of the string upon the element of string at $x = 0$.

We found that this restoring force is equal to $-T\tan\theta$ and $\tan\theta = \frac{\partial u}{\partial x}$ (the minus sign comes from the fact that is restoring)

$$\frac{dW}{dt} = F_{restoring} \times V = \frac{dE}{dt}$$

$$\frac{dE}{dt} = \left(-T\frac{\partial u}{\partial x}\right)_{x=0} \times \left(\frac{\partial u}{\partial t}\right)_{x=0} \qquad (29)$$

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Since the average value of $\sin^2\omega t$ over one complete cycle is 1/2, we can write

$$\left\langle \left(\frac{dE}{dt}\right)_I \right\rangle = \frac{1}{2}\omega k_1 T A^2 - \frac{1}{2}\omega k_1 T B^2$$

mean rate at which energy is incident on the junction

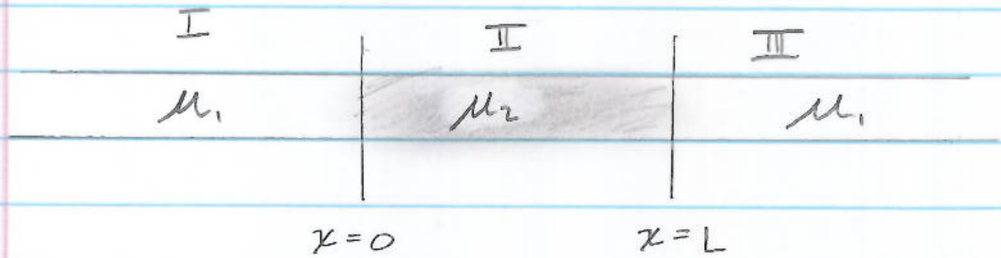
mean rate at which energy is reflected back.

Similarly

$$\left\langle \left(\frac{dE}{dt}\right)_{II} \right\rangle = \frac{1}{2}\omega k_2 C^2$$

Mean rate at which energy is supplied to the junction from left to right

Hint



$$\mu = \begin{cases} \mu_1 & x<0;\ x>L \\ \mu_2 > \mu_1 & 0<x<L \end{cases}$$

Let the wave be incident from the left

$$u_{I} = Ae^{i(\omega t - k_1 x)} + Be^{i(\omega t + k_1 x)}$$

$$u_{II} = Ce^{i(\omega t - k_2 x)} + De^{i(\omega t + k_2 x)}$$

$$u_{III} = Ee^{i(\omega t - k_1 x)}$$

The reflected intensity is $I_R = I_0 \dfrac{|B|^2}{|A|^2}$

The transmitted intensity is $I_T = I_0 \dfrac{|E|^2}{|A|^2}$

You will need to apply the continuity conditions at $x=0$ and $x=L$ to relate A, B, C, D & E.