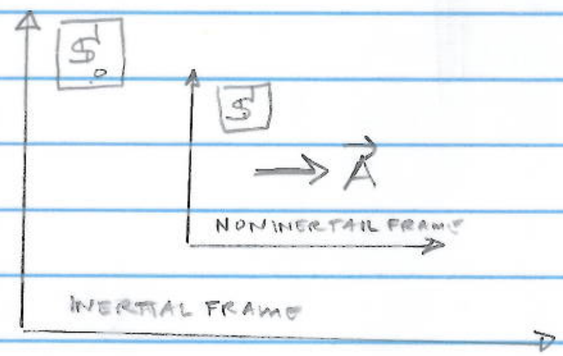
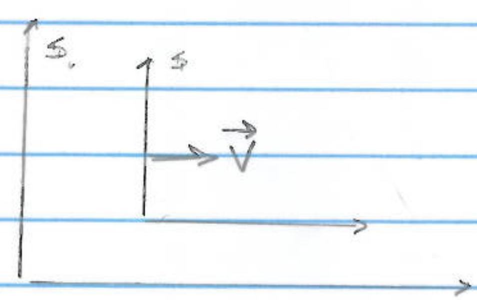
Mechanics in Noninertial Ref. Frames

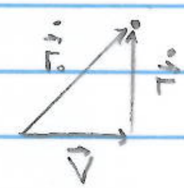
chapter 9 of Taylor.



We shall analyze the motion of a body that has an acceleration $\vec{A}$ relative to an inertial frame.

Acceleration without Rotation





Let's first look at the motion of a ball thrown straight up in frame S moving at velocity V (at a snapshot in time)

Now by NII $\quad \sum \vec{F}_i = m\ddot{\vec{r}}_0 \quad$ (in frame S_0)

From the figure $\quad \dot{\vec{r}}_0 = \dot{\vec{r}} + \vec{V}$

(ball's velocity in the inertial frame S_0)

= (ball's velocity in the moving frame S)

+ (velocity of the moving frame S wrt S_0)

$$\dot{\vec{r}} = \dot{\vec{r}}_0 - \vec{V}$$

$$\ddot{\vec{r}} = \ddot{\vec{r}}_0 - A \quad \Rightarrow$$

From N II (with the usual caveat of $m \neq m(t)$)

$\boxed{m\ddot{\vec{r}} = \vec{F} - m\vec{A}}$ ← Same form as N II but with an additional term of $-m\vec{A}$

We can continue to use N II in a NONINERTIAL FRAME provided that we add an extra forcelike term called the

INERTIAL FORCE

$\vec{F}_{Inertial} = -m\vec{A}$ ← Quite often called a "fictitious force"

Q: Why would we ever venture into accelerating frames

A:
- The Earth rotates about its axis and goes around the Sun
- Accelerating frames are quite natural frames to work in for
  - * Kinematics on our planet - long range rocket, e.g.
  - * Spinning top problems
  - * To see the behavior of bodies in the frame in which you are in.

"Fictitious Forces" are entirely real to the observer in the accelerating frame:
1) Slamming on the brakes and hitting your head on the dashboard
2) The feeling of being pushed back in your seat when the jet takes off.
3) Objects flying off the rim of a spinning body.

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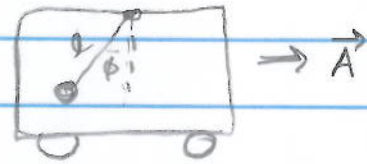
Ex Pendulum in an Accelerating Car



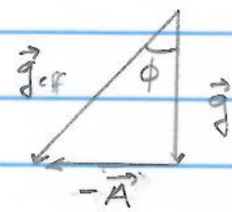
IN FRAME S_0: $\sum \vec{F} = m\ddot{\vec{r}}_0$

$$\sum \vec{F} = \vec{T} + m\vec{g}$$

Our equation of motion in Frame S becomes

$$m\ddot{\vec{r}} = \vec{T} + m\vec{g} - m\vec{A} = \vec{T} + m\underbrace{(\vec{g} - \vec{A})}_{\vec{g}_{eff}} = \vec{T} + m\vec{g}_{eff}$$

For the pendulum to remain at rest $\vec{T} = -m\vec{g}_{eff}$ i.e. $\ddot{\vec{r}} = 0$



Hence $\phi_{equilibrium} = \arctan\left(\frac{g}{A}\right)$

We know $\omega = [g/\ell]^{1/2}$ $\quad g \to g_{eff}$ and $\omega \to \left[\frac{g_{eff}}{\ell}\right]^{1/2}$

or $\omega = \left[\frac{\sqrt{g^2 + A^2}}{\ell}\right]^{1/2}$

See my notes on how I got the same results using Lagrangians.

i.e. $\vec{r} = (\ell \sin\phi + \frac{1}{2}at^2, \ell\cos\phi)$ $\quad \vec{V} = (\ell\dot{\phi}\cos\phi + at, -\ell\dot{\phi}\sin\phi)$

$$L = \frac{1}{2}mv^2 + mgy = \frac{1}{2}m(\ell^2\dot{\phi}^2 + 2at\ell\dot{\phi}\cos\phi + a^2t^2) + mg\ell\cos\phi$$

$$\frac{\partial L}{\partial \phi} = -mat\ell\sin\phi - mg\ell\sin\phi \quad \text{and} \quad \frac{\partial L}{\partial \dot{\phi}} = m\ell^2\dot{\phi} + mat\ell\cos\phi$$

And the Lagrange equation of motion $\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0$ becomes

$$\ell\ddot{\phi} = -g\sin\phi - a\cos\phi$$

We found

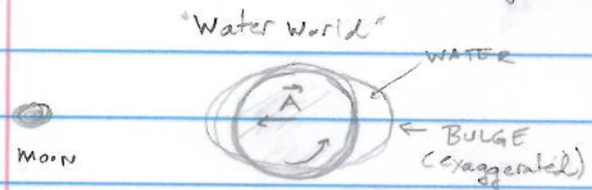
$$\hookrightarrow \ell\ddot{\phi} = -\sqrt{g^2+a^2}\,(\cos\beta\sin\phi + \sin\beta\cos\phi)$$
$$= -\sqrt{g^2+a^2}\,\sin(\phi + \beta)$$

$\ddot{\phi} = 0$ if in equilibrium and $\Rightarrow \phi = -\beta$ or $\phi = \arctan\left(\frac{a}{g}\right)$ $\quad \omega = \left[\frac{\sqrt{g^2+a^2}}{\ell}\right]^{1/2}$

If $\phi = \beta + \epsilon$ then $\ell\ddot{\epsilon} = -\sqrt{g^2+a^2}\,\sin\epsilon \Rightarrow \ddot{\epsilon} + \left[\frac{\sqrt{g^2+a^2}}{\ell}\right]\epsilon = 0$

The Tides

Our noninertial NII gives a nice explanation for the tides



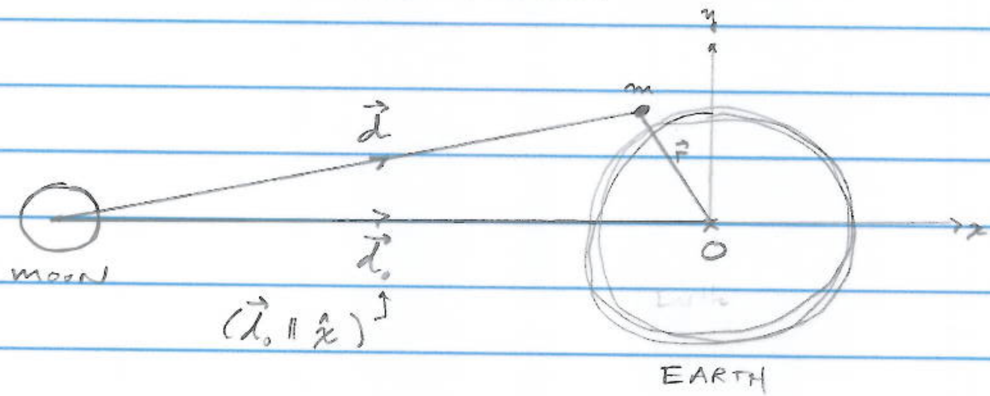
The effect of the moon's attraction is to give the entire Earth + oceans a small acceleration towards the moon.

FORCES ON any mass m near the Earth's surface.

(1) Gravitational pull $m\vec{g}$ from the Earth.

(2) Gravitational pull $-G\dfrac{M_m m}{d^2}\hat{d}$ from the Moon.

(3) Net nongravitational force $\vec{F}_{ng}$, i.e. buoyant force on a drop of sea water in the ocean.



The acceleration of origin O (the Earth's) center $\vec{A} = -GM_m \dfrac{\hat{d}_0}{d_0^2}$

Our noninertial NII $\quad m\ddot{\vec{r}} = \sum \vec{F} - m\vec{A}$

$$\sum \vec{F} = m\vec{g} + \vec{F}_{ng} + G\frac{M_m m}{d^2}\hat{d}$$

Hence

$$m\ddot{\vec{r}} = m\vec{g} + \vec{F}_{ng} + G\frac{M_m m}{d^2}\hat{d} + G\frac{M_m m}{d_0^2}\hat{d}_0$$

This can be rewritten

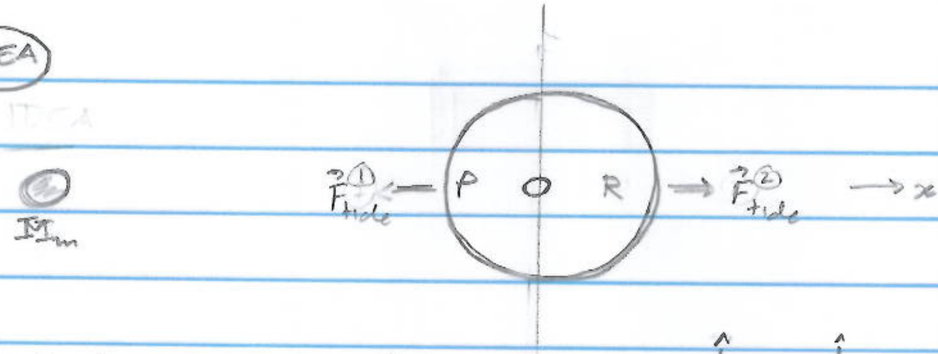
$$m\ddot{\vec{r}} = m\vec{g} + \vec{F}_{tide} + \vec{F}_{ng}$$

$\hat{d}_0 \parallel \hat{x}$

$$\vec{F}_{tide} = -GM_m m\left(\frac{\hat{d}}{d^2} - \frac{\hat{d}_0}{d_0^2}\right)$$

The tidal force is the difference between the actual force of the moon on m and the corresponding force if m were at the center of the Earth.

IDEA

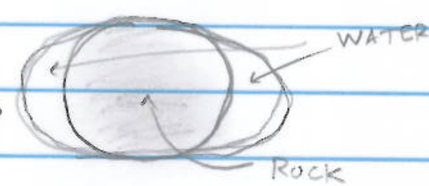


Let's examine $\vec{F}_{tide} = GM_m m\left(\underbrace{\frac{\hat{d}}{d^2}}_{(1)} - \underbrace{\frac{\hat{d}_o}{d_o^2}}_{(2)}\right)$

- At pt P, the vectors $\hat{d}$ and $\hat{d}_o$ point in the same direction. $d_o > d$ and term (1) dominates, hence $\vec{F}^{(1)}_{tide} \parallel -\hat{x}$ and points <u>towards</u> the moon

- At pt R, the vectors $\hat{d}$ and $\hat{d}_o$ again point in the same direction $d_o < d$ and term (2) dominates, hence $\vec{F}^{(2)}_{tide} \parallel +\hat{x}$ and points <u>away</u> from the moon.

AND WE GET THESE BULGES $\Rightarrow$



<u>Magnitude of the Tides</u>

First, let us consider a drop of water, on the surface of the ocean. The drop is in equilibrium (IN the EARTH FRAME) under the influence of

- $m\vec{g}$
- $\vec{F}_{tide}$
- $\vec{F}_{BUOYANCY}$ (Note: Taylor calls this "Pressure Force")

Archimedes Principle – A body is buoyed up with a force equal to the weight of the water displaced.

By the fact of Archimedes Principle the buoyant force must be normal; there can be no shearing force. And since $\vec{F}_{Buoyancy}$ must be normal and that the parcel of water is in equilibrium, it ensues that the combination of $m\vec{g}$ and $\vec{F}_{tide}$ likewise must be normal to the surface

$m\vec{g}$ and $\vec{F}_{tide}$ are conservative forces.

a) $m\vec{g} = -\nabla U_{eg}$ b) $\vec{F}_{tide} = -\nabla U_{tide}$

Earth gravity

$$\vec{F}^{①}_{tide} = -GM_m m \frac{\hat{d}}{d^2} \Rightarrow U_{①} = +GM_m m \left(\frac{1}{d}\right)$$

$$F^{②}_{tide} = -GM_m m \frac{\hat{d}_o}{d_o^2},$$

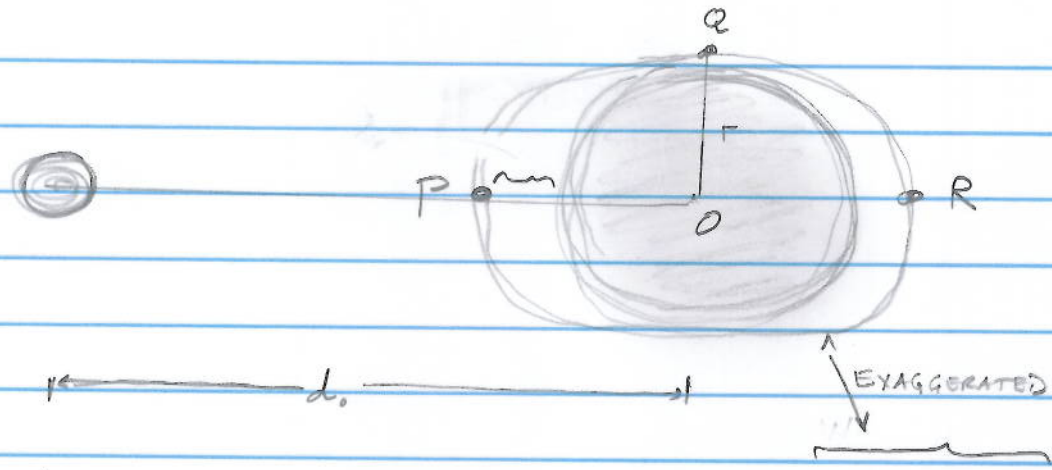
We observe that $\hat{d}_o \parallel \hat{x}$ by dint of how we set up the problem. (d_o is a fixed distance)

$$U_{②} = -\int \vec{F}\cdot d\vec{x} = \frac{GM_m m}{d_o^2}\int dx = \frac{GM_m m}{d_o^2}x$$

So adding $U_{①}$ and $U_{②}$ gives $U_{tide} = -GM_m m\left(\frac{1}{d} + \frac{x}{d_o^2}\right)$

Saying that $m\vec{g} + \vec{F}_{tide}$ is normal to the surface of the ocean can be rephrased as saying $\nabla U_{tide} + \nabla U_{eg} = \nabla(U_{eg} + U_{tide})$ is normal to the surface. This implies that $U = U_{eg} + U_{tide}$ is constant. THE SURFACE OF THE OCEAN IS AN EQUIPOTENTIAL SURFACE

9-7



We observe that $d_0 \cong 60r$. Hence $\overline{OP}$ and $\overline{OQ}$ are the same, i.e. the radius of the Earth (6400km)

Now since $U(P) = U(Q)$ ← U is constant on the surface of the ocean

$$U_{eq}(P) + U_{tide}(P) = U_{eq}(Q) + U_{tide}(Q)$$

Rewriting

(1) $$\underbrace{U_{eq}(P) - U_{eq}(Q)}_{mgh} = U_{tide}(Q) - U_{tide}(P)$$

← h is the difference between high and low tides

At point Q $d = \sqrt{d_0^2 + R_E^2}$ and $x = 0$

$$U_{tide}(Q) = -GM_m m \frac{1}{\sqrt{d_0^2 + R_E^2}}$$

$$[d_0^2 + R_E^2]^{-1/2} = \frac{1}{d_0}\Big[1 + \underbrace{\left(\frac{R_E}{d_0}\right)^2}_{\epsilon}\Big]^{-1/2} \cong \frac{1}{d_0}\left[1 - \frac{1}{2}\epsilon\right]$$

$$\cong \frac{1}{d_0}\left[1 - \frac{1}{2}\left(\frac{R_E}{d_0}\right)^2\right]$$

Therefore

(2) $$U_{tide}(Q) \cong -\frac{GM_m m}{d_0}\left(1 - \frac{1}{2}\frac{R_E^2}{d_0^2}\right)$$

At point P, we see that $d = d_0 - R_E$ and $x = -R_E$.
By means of a similar calculation (9.5) one finds

$$(3) \qquad U_{tide}(P) \cong -\frac{G M_m m}{d_0}\left(1 + \frac{R_E^2}{d_0^2}\right)$$

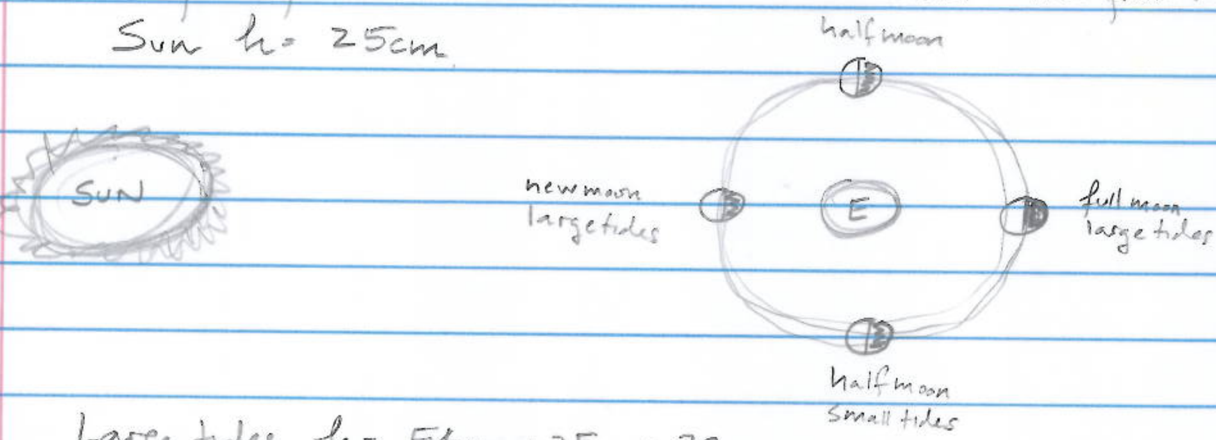
Inserting (2) and (3) into (1) gives

$$mgh = -\frac{G M_m m}{d_0}\,\frac{3}{2}\,\frac{R_E^2}{d_0^2}$$

Now $g = G\frac{M_E}{R_E^2}$. This gives

$$\boxed{h = \frac{3}{2}\,\frac{M_m}{M_E}\,\frac{R_E^4}{d_0^3}}$$

We find from the moon alone $h = 54\,cm$. And from the Sun $h = 25\,cm$



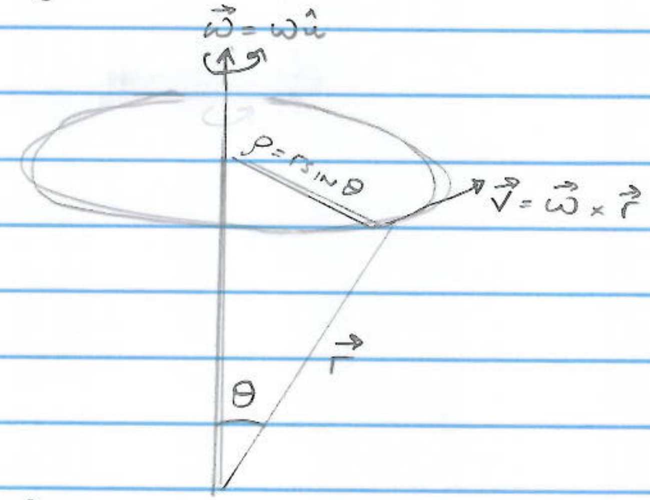
Large tides $h = 54\,cm + 25\,cm = 79\,cm$
Small tides $h = 54\,cm - 25\,cm = 29\,cm$

We have modeled our theory on "Water World". In reality there continents. These continents can block and rechannel tides through large oceans giving rise to REALLY BIG TIDES

June 5, 6, 7 → June 6, 1944 around full moon. "Allies land" at low tide (early morning) to avoid obstacles. Between 6:30am and 8:00am, the tide rose 8' @ Omaha Beach. After dropping off the men, a rising tide would help free the landing craft from the BEACHES.

9-9

Rotating Frames



$$\vec{V} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$$

A corresponding relationship for any vector fixed in a rotating body system

$$\frac{d\vec{Q}}{dt} = \vec{\omega} \times \vec{Q}$$ ← stating without proof

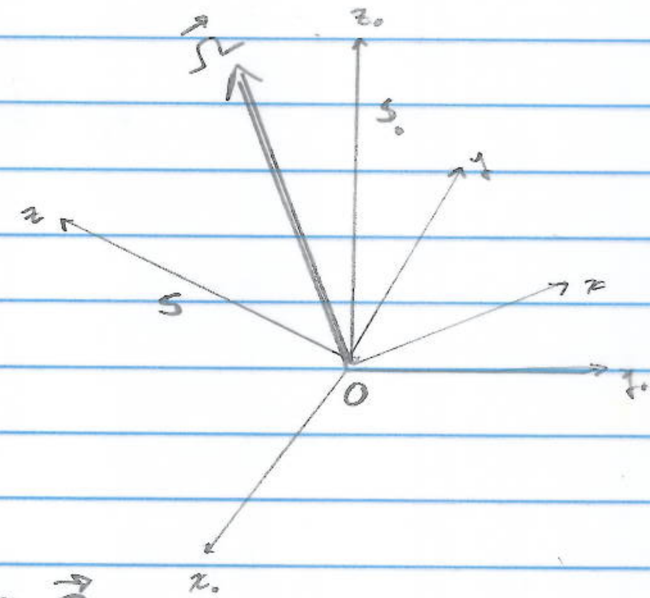
$$\frac{d\hat{e}}{dt} = \vec{\omega} \times \hat{e}$$ ← This is the unit vector $\hat{e}$ fixed in the body system. Its rate of change seen in the nonrotating system is $\underline{\vec{\omega} \times \hat{e}}$

TIME DERIVATIVES in a ROTATING FRAME

We shall consider the equations of motion for an object that is viewed from the noninertial frame S that is rotating with angular velocity Ω relative to an inertial frame S_0

Let's apply this to the Earth frame

$$\Omega = \frac{2\pi \text{ rad}}{24h \times \frac{3600s}{1h}} \cong 7.3 \times 10^{-5} \text{ rad/s}$$



The frames S_0 and S share a common origin

Frame S is rotating with angular velocity $\vec{\Omega}$ relative to S_0

Let us consider a vector $\vec{Q}$

$$\left(\frac{dQ}{dt}\right)_{S_0} = (\text{rate of change of vector } Q \text{ relative to frame } S_0)$$

$$\left(\frac{dQ}{dt}\right)_{S} = (\text{rate of change of vector } Q \text{ relative to frame } S)$$

We now wish to find how to relate these two rates of change.

$$\vec{Q} = Q_1 \hat{e}_1 + Q_2 \hat{e}_2 + Q_3 \hat{e}_3 = \sum_{i=1}^{3} Q_i \hat{e}_i$$ (In the rotating frame)

This is convenient for observors in this noninertial frame.

$$\left(\frac{d\vec{Q}}{dt}\right)_{S} = \sum_i \frac{dQ_i}{dt} \hat{e}_i$$

The unit vectors are fixed in frame S

In Frame S_0, however, $\hat{e}_i$ do vary with time

$$\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \sum \frac{dQ_i}{dt} \hat{e}_i + \sum_i Q_i \underbrace{\left(\frac{d\hat{e}_i}{dt}\right)_{S_0}}_{\vec{\Omega} \times \hat{e}_i}$$

(recall $\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$)

The second term can be rewritten

$$\sum_i Q_i \left(\frac{d\hat{e}_i}{dt}\right)_{S_0} = \sum_i Q_i (\vec{\Omega} \times \hat{e}_i) = \vec{\Omega} \times \sum Q_i \hat{e}_i = \vec{\Omega} \times \vec{Q}$$

Therefore we find the required relation:

$$\boxed{\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \left(\frac{d\vec{Q}}{dt}\right)_S + \vec{\Omega} \times \vec{Q}}$$

Newton's Second Law in a Rotating Frame

NII $\quad m\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \sum \vec{F} = \vec{F}_{Net}$

$$\left(\frac{d\vec{r}}{dt}\right)_{S_0} = \left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega} \times \vec{r}$$

Differentiating a second time

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \left(\frac{d}{dt}\right)_{S_0} \left(\frac{d\vec{r}}{dt}\right)_{S_0}$$

$$= \left(\frac{d}{dt}\right)_{S_0} \underbrace{\left[\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega} \times \vec{r}\right]}_{\vec{Q} \leftarrow \text{arbitrary vector}}$$

$$= \left(\frac{d\vec{Q}}{dt}\right)_S + \vec{\Omega} \times \vec{Q}$$

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \left(\frac{d}{dt}\right)_S \underbrace{\left[\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega} \times \vec{r}\right]}_{1} + \vec{\Omega} \times \underbrace{\left[\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega} \times \vec{r}\right]}_{2}$$

Evaluating the terms in the RHS

since $\vec{\Omega}$ = const

① $\Rightarrow \left(\frac{d^2\vec{r}}{dt^2}\right)_S + \cancel{\left(\frac{d\vec{\Omega}}{dt}\right)_S} \times \vec{r} + \vec{\Omega} \times \left(\frac{d\vec{r}}{dt}\right)_S$

② $\Rightarrow \vec{\Omega} \times \left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega} \times \vec{\Omega} \times \vec{r}$

Using the notation $\dot{\vec{r}} = \left(\frac{d\vec{r}}{dt}\right)_S$ and regrouping, we find:

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \underbrace{\ddot{\vec{r}} + 2\vec{\Omega} \times \dot{\vec{r}}}_{\text{Derivs evaluated in FRAME } S} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) =$$

By NII (in the inertial reference frame S_0)

$$m\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \vec{F} = m\ddot{\vec{r}} - 2m\dot{\vec{r}} \times \vec{\Omega} - m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$$

Hence the "noninertial NII" becomes.

$$m\ddot{\vec{r}} = \vec{F} + 2m\dot{\vec{r}} \times \vec{\Omega} + m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$$

$\vec{F} = \Sigma \vec{F}$ in the inertial frame.

- Coriolis Force:

$$\vec{F}_{cor} = 2m\dot{\vec{r}} \times \vec{\Omega}$$

- Centrifugal Force:

$$\vec{F}_{cf} = m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$$

✱ $$\boxed{m\ddot{\vec{r}} = \vec{F} + \underbrace{\vec{F}_{cor} + \vec{F}_{cf}}}$$

must add these two "fictitious" inertial forces to the net force $\vec{F}$ calculated in the inertial frame, to use NII in the rotating frame.

We shall now provide an alternate derivation of (*) using Lagrangian formalism. (see problem 9.11)

Let S be a noninertial frame with constant angular velocity $\vec{\Omega}$ relative to the inertial frame S_0. And let both have the same origin $O = O'$

First, let's find the Lagrangian. [You must evaluate T in the inertial frame]

and recall that $\vec{V}_0 = \vec{V} + \vec{\Omega} \times \vec{r}$ {Hamilton's Principle is defined in the Inertial Ref. Frame}

$$T = \frac{1}{2} m \vec{V}_0^2 = \frac{1}{2} m (\vec{V} + \vec{\Omega} \times \vec{r})^2 \Rightarrow \boxed{L = \frac{1}{2} m (\dot{\vec{r}} + \vec{\Omega} \times \vec{r})^2 - U}$$

We now shall calculate the derivatives

$$\frac{\partial L}{\partial r} = m(\dot{\vec{r}} + \vec{\Omega} \times \vec{r}) \cdot \frac{\partial}{\partial r}(\vec{\Omega} \times \overset{r\hat{r}}{\vec{r}}) - \frac{\partial U}{\partial r}$$

$$= m(\dot{\vec{r}} + \vec{\Omega} \times \vec{r}) \cdot (\vec{\Omega} \times \hat{r}) - \frac{\partial U}{\partial r}$$

$$= [m(\dot{\vec{r}} + \vec{\Omega} \times \vec{r}) \times \vec{\Omega}] \cdot \hat{r} - \frac{\partial U}{\partial r} \quad \text{(swapping "dot" and "cross")}$$

$$= \underbrace{\{[m(\dot{\vec{r}} + \vec{\Omega} \times \vec{r}) \times \vec{\Omega}] - \nabla U\}}_{} \cdot \hat{r} = \vec{F}_{NI} \cdot \hat{r} \leftarrow \boxed{\text{Force in the } \hat{r} \text{ direction}}$$

Let's find $\vec{F}_{NI}$ in the noninertial frame.

$$(1) \quad \frac{\partial L}{\partial r} = (m\dot{\vec{r}} \times \vec{\Omega} + m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega} + \overset{-\nabla U}{\vec{F}}) \cdot \hat{r}$$

$$\frac{\partial L}{\partial \dot{r}} = m(\dot{\vec{r}} + \vec{\Omega} \times \vec{r}) \cdot \hat{r} \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \underbrace{m(\ddot{\vec{r}} + \vec{\Omega} \times \dot{\vec{r}})}_{} \cdot \hat{r} \quad (2)$$

From $\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$ we obtain

$$[m\ddot{\vec{r}} + m(\vec{\Omega} \times \dot{\vec{r}}) - m\dot{\vec{r}} \times \vec{\Omega} - m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}) - \vec{F}] \cdot \hat{r} = 0$$

$$\Rightarrow \boxed{m\ddot{\vec{r}} = \vec{F} + 2m(\dot{\vec{r}} \times \vec{\Omega}) + m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}}$$

To use NII in a rotating frame (such as the frame attached to the Earth) we must introduce two INERTIAL FORCES.
We shall examine the two forces separately

We note: (for order-of-magnitude estimates)

$F_{cor} \sim mv\Omega$ $F_{cf} \sim mr\Omega^2$

v is the object's speed relative to the Earth

Ω is the ang. vel. of the Earth

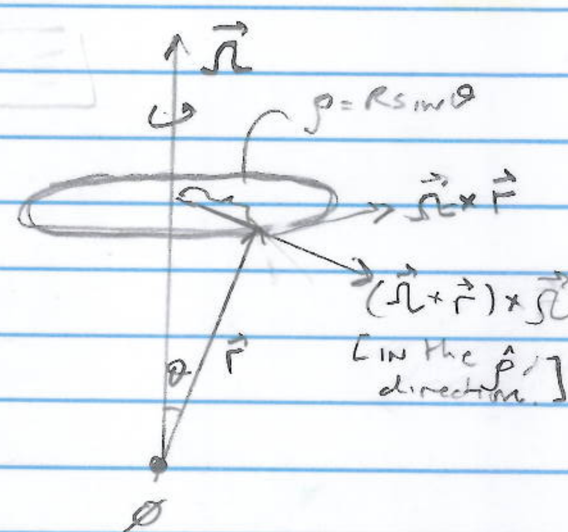
At the Earth's surface ($r = R_E$) and

$$\frac{F_{cor}}{F_{cf}} \sim \frac{v}{R_E\Omega} \sim \frac{v}{V} \qquad V \sim 1000 \text{ miles/hour}$$

Since $F_{cor} = F(v)$, for objects/projectiles with $v \ll 1000$ mi/hr the CORIOLIS FORCE (to first order) can be neglected

Let us focus on the Effects due to the

$\boxed{\text{CENTRIFUGAL FORCE}}$



$$F_{cf} = m(\vec{\Omega}\times\vec{r})\times\vec{\Omega}$$

To observers in the Earth frame there is a centrifugal force pointing radially outward from the axis of rotation

9-15

Free-fall Acceleration

In the frame relative to Earth

$$m\ddot{\vec{r}} = \vec{F}_{grav} + \vec{F}_{cf}$$

$$\vec{F}_{grav} = -\frac{GMm}{R^2}\hat{r} = m\vec{g}_o \leftarrow$$ This is the "true" acceleration due to gravity if the centrifugal force were not present.

In the Rotating Frame of Earth, the effective force becomes

$$\vec{F}_{eff} = \vec{F}_{grav} + \vec{F}_{cf} = m\vec{g}_o + m\Omega^2 R\sin\theta\,\hat{\rho}$$

The component of $\vec{g}$ in the INWARD RADIAL DIRECTION

$$\vec{g} - \vec{g}_o = (\Omega^2 R\sin\theta)\sin\theta = \Omega^2 R\sin^2\theta$$

We found earlier that $\Omega = 7.3\times10^{-5}/s$

$$\Omega^2 R = (7.3\times10^{-5}s^{-1})^2\times(6.4\times10^6 m) \doteq 0.034\ m/s^2$$

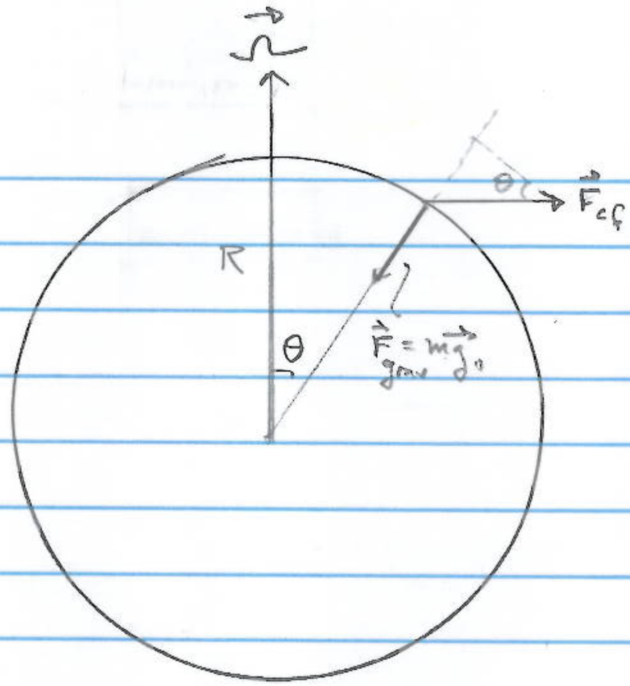
Since $|\vec{g}_o|$ is at $\sim 9.8\ m/s^2$, we see that g at the equator is 0.3% less due to the centrifugal force.

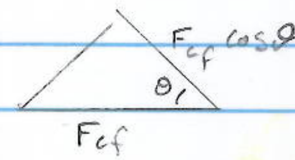
Q: Why are Rockets launched from Southern Florida instead of, say, Maine?

A: The effective g is less in Florida because of this centrifugal force.

9-16

θ: colatitude angle



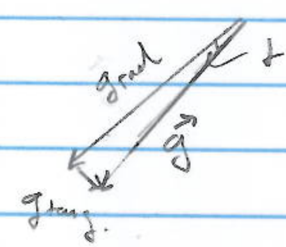


The tangential component of the effective $\vec{g}$ comes completely from the centrifugal force $\vec{F}_{cf}$

$$g_{tang} = F_{cf}\cos\theta = (\Omega^2 R \sin\theta)\cos\theta$$
$$= \Omega^2 R \frac{1}{2}\sin(2\theta)$$

g_{tang} is clearly maximum @ $\theta = 45°$
and minimum @ $\theta = 0, 90°$ (poles and equator)

$$\tan\alpha = \frac{g_{tang}}{g_{rad}}$$



α small $\Rightarrow$

$$\alpha \cong \frac{g_{tang}}{g_{rad}}$$

At $\theta = 45°$ α is maximized $\alpha = \dfrac{\Omega^2 R/2}{g_0}$

$$\alpha_{MAX} = \frac{\Omega^2 R}{2g_0} = \frac{0.034}{2(9.8)} \simeq 0.0017 \text{ rad} \sim \underline{0.1°}$$

9-17

The Coriolis Force

$$\vec{F}_{cor} = 2m\dot{\vec{r}} \times \vec{\Omega} = 2m\vec{v} \times \vec{\Omega}$$

This is a velocity-dependent force (you can compare this to $q\vec{v} \times \vec{B}$, e.g.)

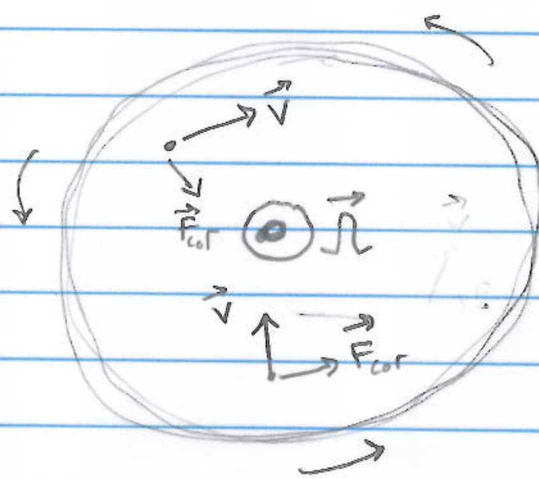
The magnitude of $\vec{F}_{cor}$ depends on the magnitudes of $\vec{v}$ and $\vec{\Omega}$ and their relative orientations.

Taking, for example, a fast baseball having $v = 50\,m/s$ with $\vec{v} \perp \vec{\Omega}$

$$a_{max} = 2v\Omega \simeq 2(50\,m/s)(7.3 \times 10^{-5}\,s^{-1}) \simeq 0.007\,m/s^2$$

Compared to $g_0 = 9.8\,m/s^2$ this is a small effect, but for long-range projectiles or for systems upon which $\vec{F}_{cor}$ acts for a long time, the Coriolis Force cannot be neglected.

Northern Hemisphere viewed from above the North Pole



9-18

As Taylor says on p. 350

It is important to bear in mind that both the Coriolis and centrifugal forces are at root kinematical effects, resulting from our insistence on using a rotating frame of reference.

One can work in an inertial frame and transform the kinematics into the noninertial frame and thereby do away with these "fictitious" forces. However, the transformation from the inertial to the noninertial frame is usually so complicated that it is easier to work in the rotating frame and live with the "fictitious" Coriolis and centrifugal forces

Free Fall and the Coriolis Force

Let us consider the effect of the Coriolis force on a freely falling object (i.e. an object falling in vacuum close to the surface of the Earth)

As the centrifugal force is usually much bigger than that of $\vec{F}_{cor}$, we must include $\vec{F}_{cf}$

Our Modified NII:

$$m\ddot{\vec{r}} = m\vec{g}_0 + \vec{F}_{cf} + \vec{F}_{cor}$$

9-19

$$\vec{F}_{cf} = m(\vec{\Omega} \times \underbrace{\vec{R}}_{|\vec{R}| = R_E}) \times \vec{\Omega}$$

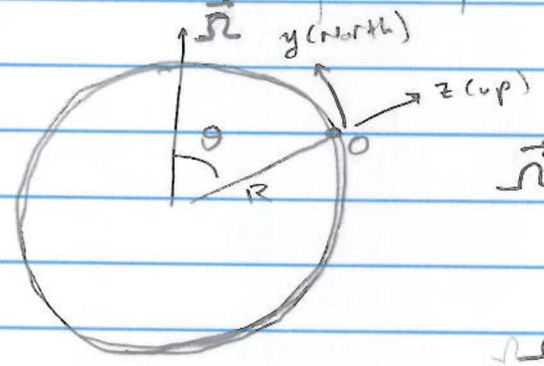
$$\vec{F}_{cor} = 2m\vec{v} \times \vec{\Omega}$$

Hence the equation of motion becomes

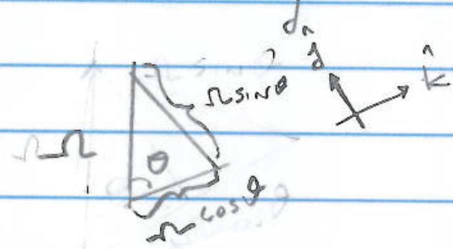
$$\ddot{\vec{r}} = \vec{g} + 2\dot{\vec{r}} \times \vec{\Omega}$$ ← Note that this expression does not involve the position $\vec{r}$, only its derivatives.

Where $\vec{g}$ is the observed acceleration due to gravity and is equal to: $\vec{g} = \vec{g}_0 + \Omega^2 R \sin\theta\, \hat{\rho}$

As the modified NII does not involve $\vec{r}$, we are free to choose our origin anywhere we like. Let's set O on the surface of the Earth



$$\vec{\Omega} = \Omega\sin\theta\,\hat{j} + \Omega\cos\theta\,\hat{k}$$



$$\dot{\vec{r}} \times \vec{\Omega} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & \Omega\sin\theta & \Omega\cos\theta \end{vmatrix} =$$

$$= (\dot{y}\Omega\cos\theta - \dot{z}\Omega\sin\theta)\hat{i} + (-\dot{x}\Omega\cos\theta)\hat{j} + (\dot{x}\Omega\sin\theta)\hat{k}$$

The equation of motion can be resolved into component form:

$$\ddot{x} = 2\Omega(\dot{y}\cos\theta - \dot{z}\sin\theta)$$
$$\ddot{y} = -2\Omega\dot{x}\cos\theta \qquad (1)$$
$$\ddot{z} = -g + 2\Omega\dot{x}\sin\theta$$

We can solve these equations through making a succession of approximations. First Ω is very, very small. To zeroeth order

$$\left.\begin{aligned}\ddot{x} &= 0\\ \ddot{y} &= 0\\ \ddot{z} &= -g\end{aligned}\right\} \text{Physic } \phi.$$

Let the object drop from rest @ $x = y = 0$ and $z = h - \frac{1}{2}gt^2$. Putting these value into equations (1), we get (i.e. $v_{x_0} = v_{y_0} = v_{z_0} = 0$ and $\dot{z} = -gt$)

$$\ddot{x} = 2\Omega g t \sin\theta \longleftarrow \text{NEW}$$
$$\ddot{y} = 0$$
$$\ddot{z} = -g$$

Integrating the x equation twice yields

$$x = \frac{1}{3}\Omega g t^3 \sin\theta \qquad (1^{st}\text{-order approximation})$$

Ex: Consider an object dropped down a 150 m shaft at the equator. Let's find the total deflection by the time it hits the bottom. We shall neglect retarding forces from the atmosphere

$$t = \left[\frac{2h}{g}\right]^{1/2} = \left[\frac{2(150\,m)}{9.8\,m/s^2}\right]^{1/2} \cong 5.5\,s.$$

$$x = \frac{1}{3}\Omega g\left(\frac{2h}{g}\right)^{3/2}$$

$$= \frac{1}{3}(7.3\times10^{-5}\,s^{-1})(9.8\,m/s^2)\left(\frac{300\,m}{9.8\,m/s^2}\right)^{3/2} \cong \boxed{4.0\,cm}$$

measureable!

9.26: The equations of motion:

$$\ddot{x} = 0 \quad \ddot{y} = 0 \quad \ddot{z} = -g$$

The initial velocity is $\vec{V}_0$. Integrating the above (from $\emptyset$)

$$V_x = V_{x0} \qquad V_y = V_{y0} \qquad V_z = -gt + V_{z0}$$

The equations of motion become [(1) from above 9-20]

$$\ddot{x} = 2\Omega(V_{y0}\cos\theta - V_{z0}\sin\theta) + 2\Omega g t \sin\theta$$

$$\ddot{y} = -2\Omega V_{x0}\cos\theta$$

$$\ddot{z} = -g + 2\Omega V_{x0}\sin\theta.$$

Integrating twice gives

$$(*)\quad \begin{cases} x(t) = V_{x0}t + \Omega(V_{y0}\cos\theta - V_{z0}\sin\theta)t^2 + \frac{1}{3}\Omega g t^3 \sin\theta \\ y(t) = V_{y0}t - \Omega(V_{x0}\cos\theta)t^2 \\ z(t) = V_{z0}t - \frac{1}{2}gt^2 + \Omega(V_{x0}\sin\theta)t^2 \end{cases}$$

9-22

9.28 A naval gun shoots a shell at colatitude θ in a direction that is α above the horizontal and due East, with muzzle speed v_0.

(a) Ignoring the Earth's rotation (and air resistance), find how long (t) the shell will be in the air and how far away it would land. (R)

We shall ignore Ω entirely and set $v_{y0} = 0$ and our equations (*)

$$x(t) = v_{x0} t \quad \text{and} \quad z(t) = v_{z0} t - \frac{1}{2} g t^2$$

The time of flight is $\frac{2v_{z0}}{g}$ and the Range $R = \frac{2 v_{x0} v_{z0}}{g}$

$$R = \frac{2v_0^2}{g} \cos\alpha \sin\alpha \quad \text{if } v_0 = 500\,\text{m/s} \text{ and } \alpha = 20°$$

$$t = \frac{2v_0 \sin\alpha}{g} = \frac{2(500\,\text{m/s}) \sin 20°}{9.8\,\text{m/s}^2} = 34.9\,\text{s}.$$

$$R = v_{x0} t = (v_0 \cos 20°)(34.9\,\text{s}) = 16.4\,\text{km}.$$

(b) From our equations (*) with $v_{y0} = 0$ $\quad \theta' = 90° - \theta$

$$y(t) = -\Omega v_{x0} \cos\theta' t^2 = -\Omega v_0 \cos\alpha \cos\theta' t^2$$

For $\theta = 50°$ $\theta' = 40°$

$$\Rightarrow y = -(7.3\times10^{-5}\,\text{rad/s})(500\,\text{m/s}) \cos 20° \cos 40° (34.9\,\text{s})^2$$
$$= -32\,\text{m} \quad (32 \text{ meters SOUTH})$$

for $\theta' \to -\theta'$ $\quad y \to -y$

(32 meter to the NORTH.

9-23

Foucault Pendulum

In S_0 (inertial frame) $m\ddot{\vec r}_0 = \sum \vec F = m\vec g + \vec T$

Now boosting to the Earth frame

$$m\ddot{\vec r} = \sum \vec F + \vec F_{cf} + \vec F_{cor}$$

$$m\ddot{\vec r} = \vec T + \underbrace{m\vec g + m(\vec\Omega \times \vec r) \times \vec\Omega}_{m\vec g} + 2m\dot{\vec r} \times \vec\Omega$$

$$m\ddot{\vec r} = \vec T + m\vec g + 2m\dot{\vec r} \times \vec\Omega$$

We set our axes such that x (east), y (north), and z (up)

direction of $-\vec g$

and for SMALL OSCILLATIONS

$T_z = T\cos\beta \simeq T$

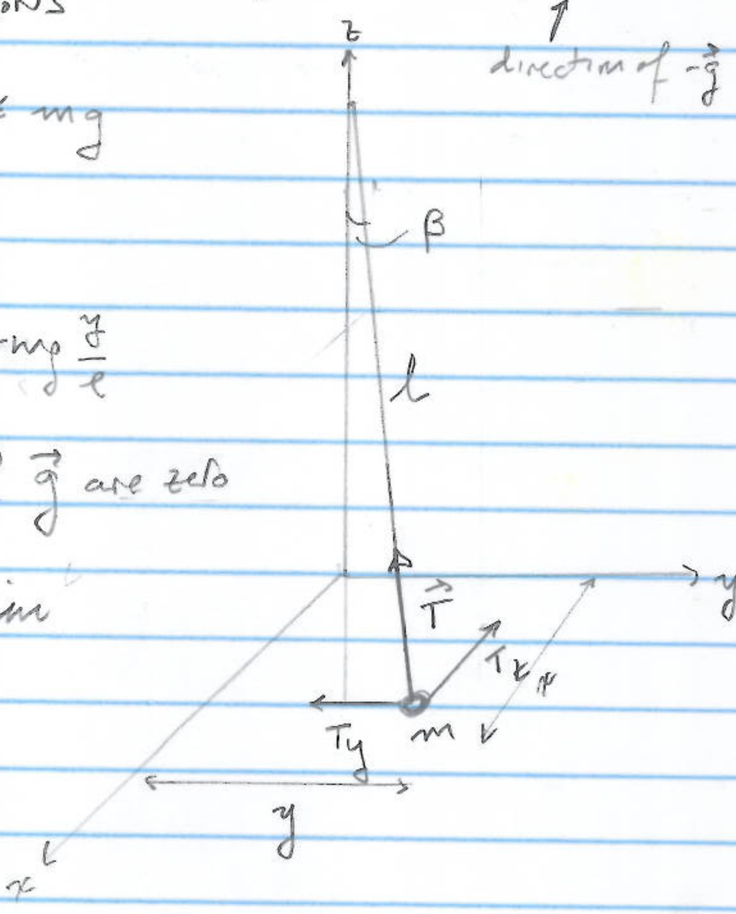
$T_z \simeq mg$ $\Rightarrow T \cong mg$

By SIMILAR TRIANGLES

$$T_x = -mg\frac{x}{\ell} \qquad T_y = -mg\frac{y}{\ell}$$

The x and y components of $\vec g$ are zero

The pendulum bob moves in the x-y plane



From p. 9-20 (eq. 9.53 in Taylor)

$$\left.\begin{aligned} \ddot{x} &= 2\Omega(\dot{y}\cos\theta - \dot{z}\sin\theta) \\ \ddot{y} &= -2\Omega\dot{x}\cos\theta \\ \ddot{z} &= -g + 2\Omega\dot{x}\sin\theta \end{aligned}\right\} \text{ with no external forces}$$

$$\left.\begin{aligned} \ddot{x} &\rightarrow \ddot{x} + \frac{1}{m}\Sigma F_x \\ \ddot{y} &\rightarrow \ddot{y} + \frac{1}{m}\Sigma F_y \end{aligned}\right\}$$

Bob is constrained to oscillate in the x-y plane (z is not interesting)

$$\ddot{x} = -g\frac{x}{\ell} + 2\Omega\dot{y}\cos\theta \qquad (\dot{z} = 0)$$

$$\ddot{y} = -g\frac{y}{\ell} - 2\Omega\dot{x}\cos\theta.$$

θ is the colatitude

Noting the $\omega_0^2 = g/\ell$ and $\Omega\cos\theta = \Omega_z$

$$\left.\begin{aligned} (1) \quad & \ddot{x} - 2\Omega_z\dot{y} + \omega_0^2 x = 0 \\ (2) \quad & \ddot{y} + 2\Omega_z\dot{x} + \omega_0^2 y = 0 \end{aligned}\right\} \text{ Coupled equations of motion}$$

Let $\eta = x + iy \qquad \dot{\eta} = \dot{x} + i\dot{y} =$

If we multiply (2) by i and add it to (1) gives

$$(3) \qquad \ddot{\eta} + 2i\Omega_z\dot{\eta} + \omega_0^2\eta = 0$$

This is a second-order linear homogeneous differential equ.

$\Rightarrow \exists$ two solns.

9-25

Let us try the trial solution $\eta(t) = e^{-i\alpha t}$ and we obtain

$$\alpha^2 - 2\Omega_z \alpha - \omega_0^2 = 0$$

$$\alpha = \Omega_z \pm \sqrt{\Omega_z^2 + \omega_0^2} = \Omega_z \pm \omega_0 \sqrt{1 + \underbrace{\frac{\Omega_z^2}{\omega_0^2}}_{\text{small}}}$$

$$\Rightarrow \alpha \simeq \Omega_z \pm \omega_0$$ [The Earth's ang vel. Ω is much smaller than ω_0

$$\Rightarrow \eta(t) = e^{-i\Omega_z t}\left(C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}\right)$$

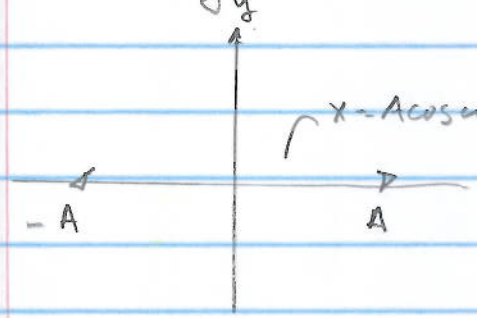
Let set the initial conditions so that we cam fix C_1 and C_2

At time $t=0$, the pendulum has been pulled to a position $x = A$ and $y = 0$ and is released from rest $V_{x_0} = V_{y_0} = 0$

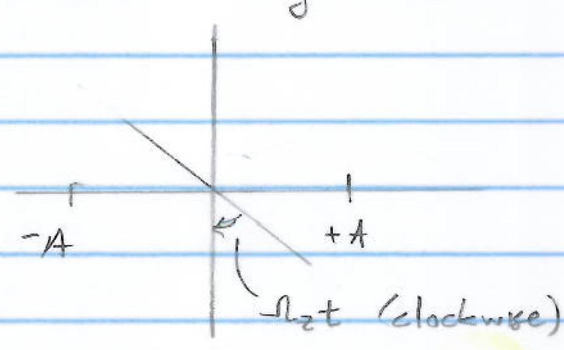
$$\eta = x + iy \Rightarrow A = C_1 + C_2$$
$$\dot{\eta} = \dot{x} + i\dot{y} \Rightarrow 0 = C_1 - C_2$$
$$\Rightarrow C_1 = C_2 = \frac{A}{2}$$

$$\Rightarrow \eta(t) = A e^{-i\Omega_z t}\left(\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}\right) = A e^{-i\Omega_z t} \cos\omega_0 t$$

Initially the pendulum swings in SHM along the x axis



@ $t=0$



$t>0$

9-26

Latitude of Pocatello $\Rightarrow$ $42.8752° \approx 42.88$

$\theta = 90° - 42.8752 \approx 47.12$

$\Omega_z = \Omega \cos(47.12)$

$= 0.68\,\Omega$

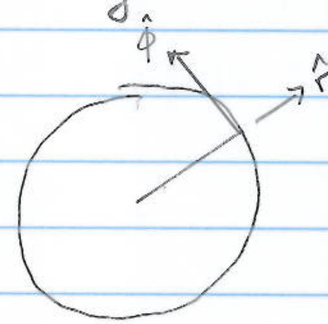
Since $\Omega = 360°/\text{day} \Rightarrow 0.68\Omega \cong 245°/\text{day}$

IN The course of 4 hours (1/6 of a day), the pendulum's plane will sweep out 40.8°. An easily observable effect.

$\mathbb{R}^2$

$\vec{r} = r\cos\phi\,\hat{x} + r\sin\phi\,\hat{y}$

$\frac{d\vec{r}}{dt} = \dot{r}(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}) + r\dot{\phi}(\sin\phi\,\hat{x} - \cos\phi\,\hat{y}) = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$



We see a trend

$\vec{r} = r\hat{r}$

$\frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\frac{d\hat{r}}{d\phi}\frac{d\phi}{dt} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$

and $\frac{d\hat{r}}{d\phi} = \frac{d}{d\phi}(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}) = \sin\phi\,\hat{x} - \cos\phi\,\hat{y} = \hat{\phi}$

Now $\hat{\phi} = \sin\phi\,\hat{x} - \cos\phi\,\hat{y} = \frac{d\hat{r}}{d\phi}$ $\boxed{\hat{\phi} = \frac{d\hat{r}}{d\phi}}$

$\frac{d\hat{\phi}}{d\phi} = \frac{d}{d\phi}(\sin\phi\,\hat{x} - \cos\phi\,\hat{y}) = -\hat{r}$

$\frac{d\hat{\phi}}{dr} = 0$

$\boxed{\hat{r} = -\frac{d\hat{\phi}}{d\phi}}$

9-27

$$\frac{d^2\hat{r}}{dt} = \frac{d}{dt}(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi})$$

$$= \ddot{r}\hat{r} + \underbrace{\dot{r}\frac{d\hat{r}}{d\phi}\dot{\phi}}_{\dot{r}\dot{\phi}\hat{\phi}} + \dot{r}\cancelto{0}{\frac{d\hat{r}}{dr}}\dot{r} + \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} + r\dot{\phi}\underbrace{\frac{d\hat{\phi}}{d\phi}}_{-\hat{r}}\frac{d\phi}{dt}$$

Collecting terms

$$\frac{d^2\vec{r}}{dt} = (\ddot{r} - r\dot{\phi}^2)\hat{r} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi}$$

SINCE $\sum F = m\frac{d^2\vec{r}}{dt^2}$

$$F_r = m(\ddot{r} - r\dot{\phi}^2)$$

$$F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi})$$

We see that $m\ddot{r} = F_r + mr\dot{\phi}^2 = F_r + \overbrace{mr\Omega^2}^{\text{cf centrifugal}}$

$\boxed{\text{INERTIAL FRAME}} \Rightarrow$ $mr\ddot{\phi} = F_\phi - 2m\dot{r}\dot{\phi} = F_\phi - \underbrace{2m\dot{r}\Omega}_{\text{cf coriolis}}$

In the inertial frame we have complicated accelerations but no "fictitious forces" and in the rotating frame it is the other way around. In general, it is more convenient to work in the ROTATING (noninertial) frame and learn to live with these "fictitious" forces.