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Let us consider the case of a general asymmetric rigid body. Let us explore the stability for small deviations from rotation about each of the spinning axis. (For example dropping a spinning book with the axis of rotation parallel to $\vec{g}$ → No torque)

For force-free asymmetrical top:

$$\boxed{\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\,\omega_2\omega_3 &= 0 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\,\omega_1\omega_3 &= 0 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\,\omega_1\omega_2 &= 0 \end{aligned}}$$

← Euler's Dynamical Equations.

Let $I_3 > I_2 > I_1$

(a) $\underline{\vec{\omega} \simeq \omega_3 \hat{x}_3}$ $\qquad \omega_3 \gg \omega_2$ & $\omega_3 \gg \omega_1$.

$\Rightarrow \omega_1$ and $\omega_2 \simeq 0$

From Euler's Dynamical Equations

$$\dot{\omega}_1 = -\frac{I_3 - I_2}{I_1}\omega_2\omega_3 \qquad (1)$$

$$\dot{\omega}_2 = -\frac{I_1 - I_3}{I_2}\omega_1\omega_3 \qquad (2)$$

$$\dot{\omega}_3 \cong 0 \Rightarrow (\omega_3 = \text{const}) \qquad (3)$$

↑ The product of ω_1 and ω_2 is very small $\simeq 0$

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$$\ddot{\omega}_1 = -\left(\frac{I_3 - I_2}{I_1}\right)\left(\cancelto{0}{\omega_2\dot{\omega}_3} + \dot{\omega}_2\omega_3\right)$$

$$= +\left(\frac{I_3 - I_2}{I_1}\right)\left(\frac{I_1 - I_3}{I_2}\right)\omega_1\omega_3^2$$

$$\ddot{\omega}_1 + \underbrace{\left[\frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}\omega_3^2\right]}_{\Omega_3^2}\omega_1 = 0$$

$$\Omega_3^2 \Rightarrow \qquad \Omega_3 = \left[\frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}\right]^{1/2}\omega_3$$

$$\Rightarrow \boxed{\omega_1 = A_3\cos(\Omega_3 t + \delta_3)} \qquad (4)$$

Now from (1)

$$\omega_3 = \left(\frac{I_1}{I_2 - I_3}\right)\frac{1}{\omega_2}\dot{\omega}_1$$

$$= -\left(\frac{I_1}{I_2 - I_3}\right)\frac{1}{\omega_2}\left[A_3\Omega_3\sin(\Omega_3 t + \delta_3)\right]$$

$$= \frac{\omega_3}{\omega_2}\Big[\left(\frac{I_1}{I_3 - I_2}\right)\left[\frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}\right]^{1/2} A_3\sin(\Omega_3 t + \delta_3)$$

$$\boxed{\omega_2 = \left[\left(\frac{I_1}{I_2}\right)\left(\frac{I_3 - I_1}{I_3 - I_2}\right)\right]^{1/2} A_3\sin(\Omega_3 t + \delta_3)} \qquad (5)$$

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$$\ddot{\omega}_1 - \Omega_2^2\,\omega_1 = 0$$

The solns to this equation is

$$\omega_1 = A_2 e^{\Omega_2 t} + A_2' e^{-\Omega_2 t}$$

WLOG, let us set the initial conditions so that $A_2' = 0$

$$\Rightarrow \omega_1(t) = A_2 e^{\Omega_2 t} \qquad (6)$$

$$\dot{\omega}_1(t) = A_2 \Omega_2 e^{\Omega_2 t} = \Omega_2 \omega_1 \qquad (6')$$

$$\ddot{\omega}_1(t) = A_2 \Omega_2^2 e^{\Omega_2 t} = \Omega_2^2 \omega_1$$

From (1)

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1}\,\omega_2\omega_3 \qquad (7)$$

Equating (6') and (7) yields

$$A_2 \Omega_2 e^{\Omega_2 t} = \frac{I_2 - I_3}{I_1}\,\omega_2\omega_3$$

$$A_2 e^{\Omega_2 t}\left[\frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}\right]^{1/2} \cancel{\omega_2} = -\frac{I_3 - I_2}{I_1}\,\cancel{\omega_2}\,\omega_3$$

We see that about the I_3 axis (Maximum Principal Axis), the motion is stable and periodic. The vector $\vec{\omega}$ moves counterclockwise (looking down from the positive $\hat{x}_3$-axis) and traces out a small ellipse about this $\hat{x}_3$-axis.

(b) $\underline{\vec{\omega} \simeq \omega_2 \hat{x}_2}$ $\quad \omega_2 \gg \omega_3$ & $\omega_2 \gg \omega_1$

$\Rightarrow$ ω_1 and $\omega_3 \simeq 0$

$$\dot{\omega}_1 = -\left(\frac{I_3 - I_2}{I_1}\right)\omega_2\omega_3$$

$$\dot{\omega}_2 \cong 0 \Rightarrow (\omega_2 \cong \text{const})$$ ← The product of ω_1 and ω_3 is very small.

$$\dot{\omega}_3 = -\left(\frac{I_2 - I_1}{I_3}\right)\omega_1\omega_2$$

$$\ddot{\omega}_1 = -\left(\frac{I_3 - I_2}{I_1}\right)\left[\cancelto{0}{\dot{\omega}_2}\omega_3 + \omega_2\dot{\omega}_3\right]$$

$$= +\left[\frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}\right]\omega_2^2\omega_1$$

$$\ddot{\omega}_1 - \underbrace{\left[\frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}\omega_2^2\right]}_{\Omega_2^2}\omega_1 = 0$$

$$\Omega_2 = \left[\frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_2}\right]^{1/2}\omega_2 > 0$$ and is Real.

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$$\boxed{\omega_3(t) = -\left[\frac{I_1}{I_3}\frac{(I_2-I_1)}{(I_3-I_1)}\right]^{1/2} A_2 e^{\Omega_2 t}}$$

$$\boxed{\omega_1(t) = A_2 e^{\Omega_2 t}}$$

Tumbling behavior.

We see that if $\vec{\omega}$ is nearly parallel to the I_2-axis, the solution is of an exponential character. ω_1 and ω_3 will not remain small and ω_2 will not remain constant nor periodic.

(c) $\vec{\omega} \simeq \omega_1 \hat{x}_1$ $\qquad \omega_1 \gg \omega_3$ and $\omega_1 \gg \omega_2$

$\Rightarrow \omega_2$ and $\omega_3 \simeq 0$

$\omega_1 = \text{const.}$

$$\dot{\omega}_2 = -\frac{I_1 - I_3}{I_2}\omega_1\omega_3 \qquad (8)$$

$$\dot{\omega}_3 = -\frac{I_2 - I_1}{I_3}\omega_1\omega_2 \qquad (9)$$

$$\ddot{\omega}_2 = -\frac{I_1 - I_3}{I_2}\omega_1\dot{\omega}_3 = \frac{(I_1-I_3)(I_2-I_1)}{I_2 I_3}\omega_1^2\omega_2$$

$$\ddot{\omega}_2 + \underbrace{\left[\left(\frac{(I_3-I_1)(I_2-I_1)}{I_2 I_3}\right)\omega_1^2\right]}_{\Omega^2}\omega_2 = 0$$

Hence

$$\boxed{\omega_2 = A_1 \cos(\Omega_1 t + \delta_1)}$$

$$\dot{\omega}_2 = A_1 \Omega_1 \sin(\Omega_1 t + \delta_1) \qquad (9)$$

Equating (8) and (9) yields

$$A_1 \Omega_1 \sin(\Omega_1 t + \delta_1) = \frac{I_3 - I_1}{I_2} \omega_1 \omega_3$$

$$\left[\frac{(I_3 - I_1)(I_2 - I_1)}{I_2 I_3} \omega_1^2\right]^{1/2} A_1 \sin(\Omega_1 t + \delta_1) = \frac{I_3 - I_1}{I_2} \omega_1 \omega_3$$

$$\boxed{\omega_3 = \left[\frac{I_2}{I_3} \; \frac{I_2 - I_1}{I_3 - I_1}\right]^{1/2} A_1 \sin(\Omega_1 t + \delta_1)}$$

We can therefore conclude that the rotation about the axes of minimum and maximum moments of inertial (Principal Axes I_1 and I_3) is stable, while rotation about the intermediate axis is unstable.

Q.E.D.

Motion of a symmetric top with one point fixed.

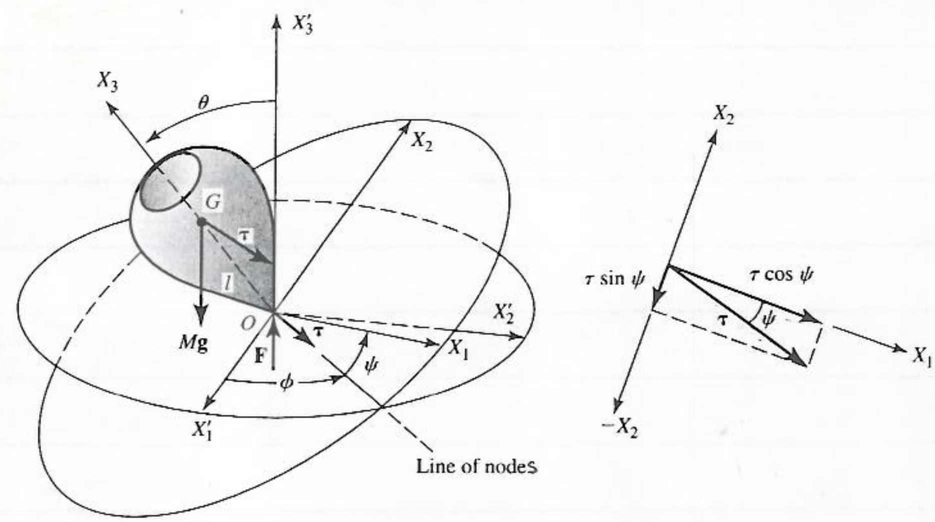

Figure 13.12 Heavy symmetrical top with one point fixed.

The symmetric top, represented above, is a rigid body for which $I_1 = I_2$. It pivots around a fixed point O that lies on the axis of symmetry a distance l from the center of mass G which also lies on the axis of symmetry. The only external forces are the forces of constraint at O and the force of gravity. The kinetic energy is given by

$$T = \frac{1}{2}\sum_i I_i \omega_i^2 = \frac{1}{2} I_1 (\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2$$

We have in the BODY COORDINATE SYSTEM [cf. EULER ANGLES]

$$\omega_1^2 = (\dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi)^2$$
$$= \dot\phi^2 \sin^2\theta \sin^2\psi + 2\dot\phi\dot\theta \sin\theta \sin\psi \cos\psi + \dot\theta^2 \cos^2\psi$$

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$$\omega_2^2 = (\dot\phi \sin\theta\cos\psi - \dot\theta\sin\psi)^2$$
$$= \dot\phi\sin^2\theta\cos^2\psi - 2\dot\phi\dot\theta\sin\theta\sin\psi\cos\psi + \dot\theta^2\sin^2\psi$$

so that

$$\omega_1^2 + \omega_2^2 = \dot\phi^2\sin^2\theta + \dot\theta^2$$

and

$$\omega_3 = (\dot\phi\cos\theta + \dot\psi)^2$$

Therefore

$$T = \frac{1}{2}I_1(\dot\phi^2\sin^2\theta + \dot\theta^2) + \frac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2$$

Because the potential energy $U = Mgl\cos\theta$, the Lagrangian $L = T - U$ Becomes

$$L = T - U = \frac{1}{2}I_1(\dot\phi^2\sin^2\theta + \dot\theta^2) + \frac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2 - Mgl\cos\theta$$

The coordinates ψ and ϕ are ignorable (or cyclic)

$$\frac{dp_\psi}{dt} = \frac{\partial L}{\partial \psi} = 0$$

$$\frac{\partial p_\phi}{\partial t} = \frac{\partial L}{\partial \phi} = 0$$

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The momenta conjugate to these coordinates are therefore constants of the motion (i.e. $p_\phi = \text{const}$ and $p_\psi = \text{const}$)

$$(1) \quad p_\phi = \frac{\partial L}{\partial \dot\phi} = I_1 \dot\phi \sin^2\theta + I_3 \cos\theta (\dot\psi + \dot\phi \cos\theta) = \text{constant}$$

$$(2) \quad p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3 (\dot\psi + \dot\phi \cos\theta) = \text{constant}$$

We observe that these cyclic coordinates are <u>angles</u>. The conjugate momenta, then, are <u>ANGULAR MOMENTA</u>

Because the symmetric top is in a uniform gravitational force field, which is CONSERVATIVE, the total energy is a constant of the motion

$$\frac{dE}{dt} = -\frac{\partial L}{\partial t} = 0$$

$$(3) \quad E = T + U = \frac{1}{2} I_1 (\dot\phi^2 \sin^2\theta + \dot\theta^2) + \frac{1}{2} I_3 (\dot\phi \cos\theta + \dot\psi)^2 + Mgl\cos\theta = \text{constant}$$

From equation (2)

$$p_\psi = I_3 (\dot\psi + \dot\phi \cos\theta) = I_3 \omega_3 = \text{constant.} \Rightarrow \omega_3 = \frac{p_\psi}{I_3} \quad (4)$$

We observe then that the second term in equ. (3) is

$$\frac{1}{2} I_3 (\dot\phi \cos\psi + \dot\psi)^2 = \frac{1}{2} I_3 \omega_3^2 = \frac{1}{2} \frac{p_\psi^2}{I_3} \quad (5)$$

From (1)

using (4)

$$I_1 \dot{\phi} \sin^2\theta = p_\phi - I_3 \cos\theta \, (\omega_3) = p_\phi - p_\psi \cos\theta$$

$$\Rightarrow \quad \dot{\phi} = \frac{p_\phi - p_\psi \cos\theta}{I_1 \sin^2\theta} \qquad (6)$$

inserting (5) and (6) into (3) yields

$$(7) \quad E = \tfrac{1}{2} I_1 \dot{\theta}^2 + \frac{p_\phi - p_\psi \cos\theta}{2 I_1 \sin^2\theta} + \frac{p_\psi^2}{2 I_3} + Mg\ell\cos\theta$$

We can now solve the problem by means of the energy method. Now both E and p_ψ are constants of the motion. If we set

$$(7') \qquad E' = E - \frac{p_\psi^2}{2 I_3}$$ ← which is also const.

From (7) we obtain

$$E' = \tfrac{1}{2} I_1 \dot{\theta}^2 + U'(\theta)$$

where

$$U'(\theta) = \frac{p_\phi - p_\psi \cos\theta}{2 I_1 \sin^2\theta} + Mg\ell\cos\theta$$

and $U'(\theta)$ is called the EFFECTIVE POTENTIAL.

Therefore

$$\dot{\theta} = \left\{ \frac{2}{I_1} [E' - U'(\theta)] \right\}^{1/2}$$

And θ is given, in principle, by computing the integral

$$\left(\frac{I_1}{2}\right)^{1/2} t = \int_{\theta_i}^{\theta} \frac{d\theta}{[E' - U'(\theta)]^{1/2}}$$

and solving for $\theta(t)$. Here θ_i is the initial value of θ.

Once $\theta(t)$ is known equ's (1) and (2) can be solved for $\dot{\psi}$ and $\dot{\phi}$ and integrated to give

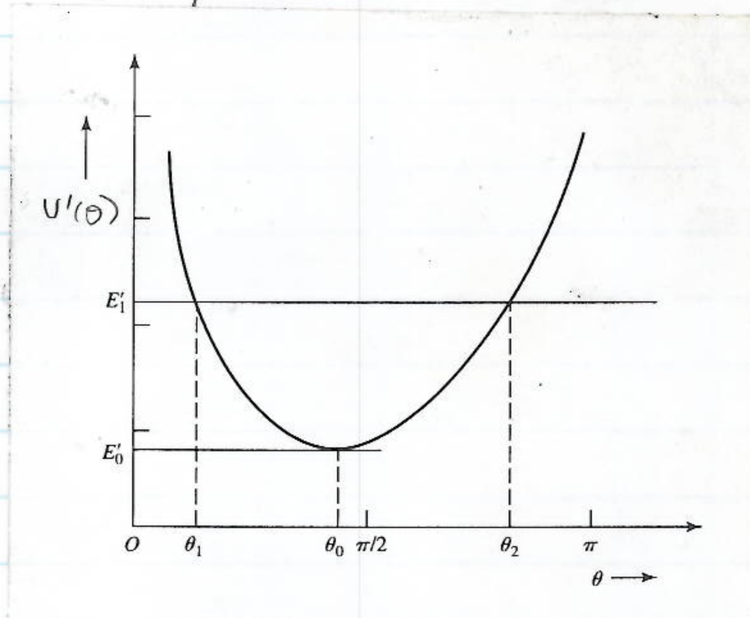
$$\psi = \psi(t) \quad \text{and} \quad \phi = \phi(t)$$

We observe again that $p_\psi = I_3 \omega_3$ so that ω_3 is a constant of the motion as well.

The 'torque' associated with the effective potential energy $U'(\theta)$ is

$$(8) \qquad \Gamma' = -\frac{\partial U'}{\partial \theta} = + mgl \sin\theta - \frac{(p_\phi - p_\psi \cos\theta)(p_\psi - p_\phi \cos\theta)}{I_1 \sin^3\theta}$$

Inspection of equ (8) shows that, in general (if $p_\psi \neq p_\phi$), the "torque" $U'(\theta)$ is positive for $\theta \doteq 0$ and negative for $\theta \simeq \pi$. For $U'(\theta)$ to be physical θ must be in the range of $0 \leq \theta \leq \pi$, as depicted in the figure below.



For any energy value $E' = E'_1$, the motion is limited to travel between the two extreme points $\theta = \theta_1$ and $\theta = \theta_2$. This means that the symmetrical axis Ox_3 (BODY FRAME) of the rotating top can vary its inclination θ 'wrt the vertical between $\theta_1 \leq \theta \leq \theta_2$. If the energy of the top is such that $E' = E'_0 = U'_{MIN}$, the value of θ will be limited to a SINGLE VALUE of inclination θ_0. This is an interesting special case of steady precession wherein the axis of the gyroscope or heavy top describes a right circular cone about the vertical (x'_3-axis).

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We can evaluate the value of θ_0 by setting the derivative of the effective potential $U'(\theta)$ equal to zero at $\theta = \theta_0$.

$$-\left.\frac{\partial U'}{\partial \theta}\right|_{\theta=\theta_0} = +Mgl\sin\theta_0 - \frac{(p_\phi - p_\psi\cos\theta_0)(p_\psi - p_\phi\cos\theta_0)}{I_1\sin^3\theta_0} = 0 \qquad (9)$$

We shall define

$$\beta \equiv p_\phi - p_\psi\cos\theta_0$$

and rewrite (9) as

$$(\cos\theta_0)\beta^2 - (p_\psi\sin^2\theta_0)\beta + (MglI_1\sin^4\theta_0) = 0$$

This is quadratic in β and $\beta \equiv p_\phi - p_\psi\cos\theta_0$ has two solutions.

$$p_\phi - p_\psi\cos\theta_0 = \frac{p_\psi\sin^2\theta_0}{2\cos\theta_0}\left[1 \pm \sqrt{1 - \frac{4MglI_1\cos\theta_0}{p_\psi^2}}\right] \qquad (10)$$

Because the quantity $\beta \equiv p_\phi - p_\psi\cos\theta_0$ MUST BE A REAL QUANTITY, the radicand in equ. (10) must be positive. This implies for $\theta_0 \leq \frac{\pi}{2}$

$$p_\psi^2 \geq 4MglI_1\cos\theta_0$$

Because $p_\psi = I_3\omega_3$ (cf. equs (2) & (4))

$$\omega_3 \geq \frac{2}{I_3}\sqrt{MglI_1\cos\theta_0} \qquad (11)$$

If $E' = U'(\theta_0)$, the axis of the top spins uniformly at an angle θ_0 wrt the vertical, and with uniform angular velocity (cf. equ. (6))

$$\dot{\phi}_0 = \frac{p_\phi - p_\psi \cos\theta_0}{I_1 \sin^2\theta_0} \qquad (12)$$

For $\omega_3 > \omega_{min}$, where $\omega_{min} = \frac{2}{I_3}\sqrt{Mgl I_1 \cos\theta_0}$ there are two roots for $\beta \equiv p_\phi - p_\psi \cos\theta_0$ (cf (10)) and hence two possible values of $\dot{\phi}_0$ in equ. (12).

$\dot{\phi}_{0(+)} \rightarrow$ Fast precession

$\dot{\phi}_{0(-)} \rightarrow$ Slow precession

For $\omega_3 \gg \omega_{min}$ the p_ψ is large and the second term in the radicand of equ. (10) is small. We may then expand the radical by means of the binomial expansion and retaining only the first nonvanishing terms, we find:

$$\dot{\phi}_{0(+)} \cong \frac{I_3}{I_1}\frac{\omega_3}{\cos\theta_0}$$

$$\dot{\phi}_{0(-)} \cong \frac{Mgl}{I_3\omega_3}$$

It is the slow precessional angular velocity that is usually observed in GYROSCOPES

Thus, for the symmetry axis at $\theta = \theta_0$ for $\theta_0 < \frac{\pi}{2}$, the top rotates about the symmetry axis at the angular frequency of ω_3 and the symmetry axis can precess about the fixed axis with two possible angular frequencies $\dot{\phi}_0$.

For the case in which $\theta_0 > \pi/2$, wherein the fixed tip of the top is at a positive ABOVE the center of mass of the top, the symmetrical top is hanging with its axis below the horizontal. One finds that the values of $\dot{\phi}_0(+)$ and $\dot{\phi}_0(-)$ have opposite signs. That is, for $\theta_0 > \pi/2$, the fast precession $\dot{\phi}_{0f}$ is in the same direction as that for $\theta_0 < \pi/2$, while for the slow precession $\dot{\phi}_{0s}$ travels in the opposite sense.

If the top is spinning sufficiently fast and is in the vertical position, the axis of the top will remain in the vertical direction. This condition is called **sleeping**, and the top is a sleeping top. If the top slows down due to friction or other causes, the top will undergo a

NUTATION

or oscillation of the axis of the top in the θ-direction as it precesses.

θ Motion - NUTATION

Study of the $U'(\theta)$ vs θ figure on p. 11-78 shows the more general motion involving a NUTATION: the top precesses monotonically about the x_3'-axis, and the x_3- (or symmetry) axis oscillates between $\theta = \theta_1$ and $\theta = \theta_2$

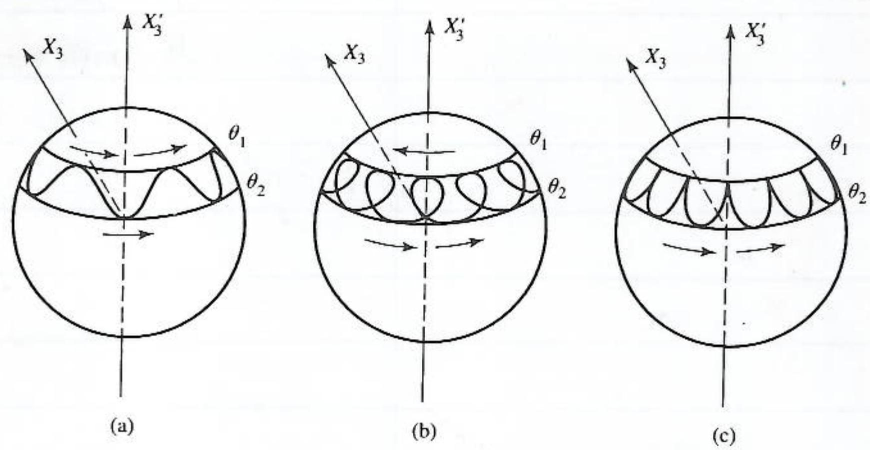

Figure 13.14 θ motion (nutation), the motion of the symmetrical axis limited between $\theta_1 < \theta < \theta_2$. The diagrams show the path of a body symmetry axis as projected on a unit sphere in a fixed system. (a) monotonic precession, (b) looping motion, and (c) cusplike motion of the symmetry axis about a fixed axis.

and must satisfy the equation

$$E' = \frac{(p_\phi - p_\psi \cos\theta)^2}{2I_1 \sin^2\theta} + IIgl\cos\theta, \qquad (13)$$

where p_ϕ, p_ψ, and E' are determined from the initial conditions. If we multiply the above by $\sin^2\theta$ (i.e. $1-\cos^2\theta$), it becomes a cubic equation in $\cos\theta$. We see from the figure on p. 11-78 that there must be two real roots $\cos\theta_1$ and

$\cos\theta_2$ between -1 and $+1$. The third root will be greater than $+1$ and therefore lie outside the physical range. (N.B. In the case of uniform precession, the two physical roots coincide $\cos\theta_1 = \cos\theta_2 = \cos\theta_0$.)

During nutation, the precession velocity varies in accordance to equ. (6)

$$\dot{\phi} = \frac{p_\phi - p_\psi \cos\theta}{I_1 \sin^2\theta} \qquad (6)$$

If $|p_\phi| < |p_\psi|$, we can define an angle θ_3:

$$\cos\theta_3 = \frac{p_\phi}{p_\psi}$$

For $\theta > \theta_3$, $\dot{\phi}$ has the same sign as ω_3, and for $\theta < \theta_3$, $\dot{\phi}$ has the opposite sign.

Let us investigate an important special case. Let the top, spinning about its axis with angular velocity ω_3, be held with its symmetry axis initially at rest at an angle θ_1 and then be released. Initially, we have

$$\theta = \theta_1, \quad \dot{\theta} = 0, \quad \dot{\phi} = 0, \quad \dot{\psi} = \omega_3$$

We substitute these initial conditions into equ's (1), (2), and (13) to find.

$$p_\psi = I_3\omega_3, \quad p_\phi = I_3\omega_3\cos\theta_1, \quad E' = Mgl\cos\theta_1,$$

In this case, we see that $\theta_3 = \theta_1$, and the motion is shown in part (c) of the figure on 11-82. - the nutation is a cusplike motion of the symmetry axis about a fixed axis.

The effective potential $U'(\theta)$ becomes
(see p. 11-76)

$$U'(\theta) = \frac{I_3^2\omega_3^2}{2I_1}\left[\frac{(\cos\theta_1 - \cos\theta)^2}{\sin^2\theta} + \alpha\cos\theta\right],$$

where

$$\alpha = \frac{2I_1 Mgl}{I_3^2\omega_3^2}$$

The turning points for the nutation are the roots of equ. (13), which becomes after we multiply through by $\sin^2\theta$

$$(\cos\theta_1 - \cos\theta) - \alpha(\cos\theta_1 - \cos\theta)(1-\cos^2\theta) = 0$$

The roots are:

$$\cos\theta = \cos\theta_1 ,$$

$$\cos\theta = \frac{1}{2\alpha}\left[1 \pm \sqrt{1 - 4\alpha\cos\theta_1 + 4\alpha^2}\right].$$

The angle θ_2 is given by the second formula using the minus sign in the bracket expression. The plus sign gives a root for $\cos\theta$ greater than $+1$.

Let us consider the case of a rapidly spinning top; that is, when $\alpha \ll 1$. We have then

$$\cos\theta_2 = \frac{1}{2\alpha}\left[1 - (1 - x)^{1/2}\right] \qquad x = 4\alpha\cos\theta_1 - 4\alpha^2$$

$$(1-x)^{1/2} = 1 - \frac{1}{2}x - \frac{1}{8}x^2$$

$$= 1 - \frac{1}{2}(4\alpha\cos\theta_1 - 4\alpha^2) - \frac{1}{8}(4\alpha\cos\theta_1 - 4\alpha^2)^2$$

$$= 1 - 2\alpha\cos\theta_1 + 2\alpha^2 - 2\alpha^2\cos^2\theta_1 + O(\alpha^3)$$

$$= 1 - 2\alpha\cos\theta_1 + 2\alpha^2(\sin^2\theta_1) + O(\alpha^3)$$

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Hence

$$\cos\theta_2 \cong \cos\theta_1 - \alpha \sin^2\theta_1 \,.$$

The angle θ_2 is only slightly greater than θ_1, and the amplitude of nutation is proportional to α