- Cole -

# VECTORS

<u>Theorem</u> The components of the sum of a number of vectors are equal to the sums of the components of the vectors.

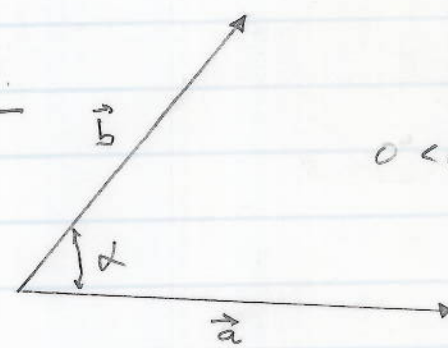
<u>Proof</u>:

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

Addition of both sides lead to the condition:

$$\vec{a}+\vec{b} = (a_1+b_1)\hat{i} + (a_2+b_2)\hat{j} + (a_3+b_3)\hat{k} \qquad \text{Q.E.D.}$$

## The scalar product.




$0 < \alpha < \pi$

$$\vec{a}\cdot\vec{b} = ab\cos\alpha$$

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

<u>Thm 1</u> $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (commutative)

<u>Proof</u>

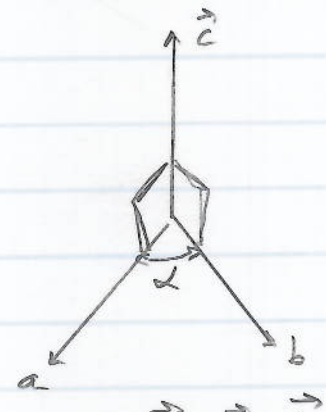
$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$
$$\vec{b}\cdot\vec{a} = b_1a_1 + b_2a_2 + b_3a_3$$

Since $a_1b_1 = b_1a_1$ the truth of the thm follows forthwith

(i) $\vec{c}$ is $\perp$ to both $\vec{a}$ and $\vec{b}$

(ii) right hand rule $\vec{a}$ to $\vec{b}$ $\Rightarrow$ thumb points in direction of $\vec{c}$.

(iii) $c = ab\sin\alpha$

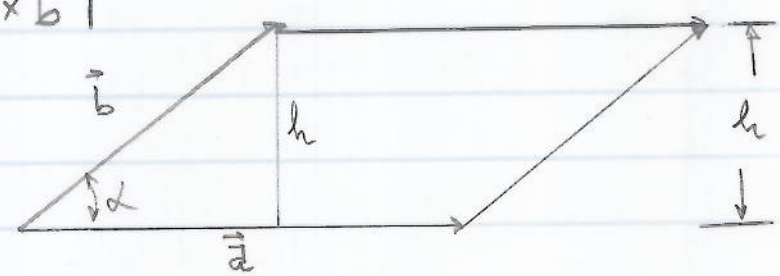



$\vec{c} = \vec{a} \times \vec{b}$

<u>Thm 1</u> The area $A$ of the parallelogram with vectors $\vec{a}$ and $\vec{b}$ forming adjacent edges is given by the relation

$$A = |\vec{a} \times \vec{b}|$$

<u>Proof</u>




If $h$ is the perpendicular distance from the terminus of $\vec{b}$ to the line of action of $\vec{a}$, then $A = ah$. But $h = b\sin\alpha$. Hence

$$A = ab\sin\alpha$$
$$= |\vec{a} \times \vec{b}| \qquad \text{Q.E.D}$$

**Thm 2** The scalar product is distributive

$$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$$

If $\vec{a}$ and $\vec{b}$ are perpendicular, then $\vec{a}\cdot\vec{b}=0$
However, if it is given that $\vec{a}\cdot\vec{b}=0$, it does not necessarily follow that $\vec{a}\perp\vec{b}$. It can only be said that at least one of the following is true:
$a=0$; $b=0$, $\vec{a}\perp\vec{b}$. Similarly, if it is given that

$$\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}$$

it does not necessarily follow that $\vec{b}=\vec{c}$.

$$\Rightarrow \quad \vec{a}\cdot(\vec{b}-\vec{c})=0$$

Hence it can be said that only one of the following is true:
1. $a=0$
2. $\vec{b}=\vec{c}$
3. $\vec{a}\perp(\vec{b}-\vec{c})$

N.B: $\vec{a}\cdot\vec{a}=a^2$
$\hat{i}\cdot\hat{i}=1$
$\hat{i}\cdot\hat{j}=0$
$\hat{i}\cdot\hat{k}=0$
$\hat{j}\cdot\hat{k}=0$

## $\boxed{\text{The vector product}}$

Let us again consider two vectors $\vec{a}$ and $\vec{b}$, the smallest nonnegative angle between them being $\alpha$. $(0°<\alpha<180°)$. The vector product of $\vec{a}$ and $\vec{b}$ is a third vector $\vec{c}$ defined in terms of $\vec{a}$ by the following three conditions:

It can be shown (or we can take as the definition):

$$\vec{a}\times\vec{b} = (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

or, in determinate form

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

<u>Thm 2</u> $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$ The vector product is NOT commutative

* Assign the proof of this a HW problem.

<u>Thm 3</u> The vector product is distributive

$$\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a} + \vec{c}$$

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N.B. $\vec{a}\times\vec{a} = 0$

$$\hat{i}\times\hat{i} = 0 \qquad \hat{i}\times\hat{j} = \hat{k} \qquad \hat{i}\times\hat{k} = -\hat{j}$$

$$\hat{j}\times\hat{i} = -\hat{k} \qquad \hat{j}\times\hat{j} = 0 \qquad \hat{j}\times\hat{k} = \hat{i}$$

$$\hat{k}\times\hat{i} = \hat{j} \qquad \hat{k}\times\hat{j} = -\hat{i} \qquad \hat{k}\times\hat{k} = 0$$

# Multiple products of Vectors

Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be any three vectors. The expression

$$\vec{a}\cdot(\vec{b}\times\vec{c})$$

is a scalar, and is called a <u>scalar triple product</u> of $\vec{a}$, $\vec{b}$, and $\vec{c}$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)$$

or

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

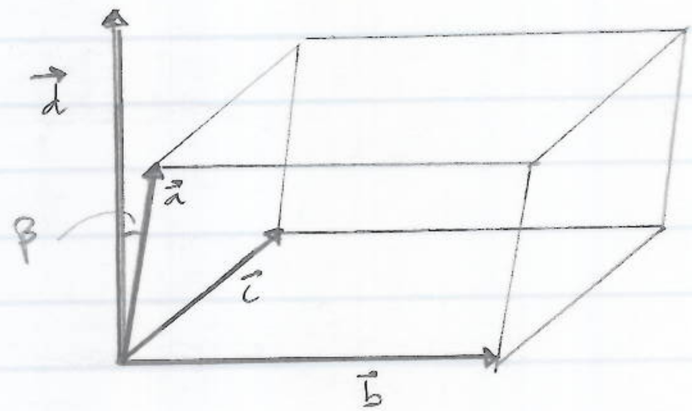
<u>Thm 1</u> The permutation theorem for scalar triple products. If the vectors in a scalar triple product are subjected to an odd number of permutations, the value of this product is changed only in sign; and if the number of permutations is even the value of the product is not changed.

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$$
$$= -\vec{c}\cdot(\vec{b}\times\vec{a}) = -\vec{a}\cdot(\vec{c}\times\vec{b})$$
$$= -\vec{b}\cdot(\vec{a}\times\vec{c})$$

Thm 2 The volume $V$ of the parallelepiped with the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ forming adjacent edges is given by the relation:

$$V = |\vec{a}\cdot(\vec{b}\times\vec{c})|$$

Proof:




Let $\vec{d} = \vec{b}\times\vec{c}$. Then $d = |\vec{b}\times\vec{c}|$ and by an earlier theorem the area of the parallelogram forming the base of the parallelepiped is the $d$. Hence $V = hd$ where $h$ is the altitude of the parallelepiped. But $\vec{d}$ is perpendicular to the base, and if $\beta$ is the angle between $\vec{a}$ and $\vec{d}$, then $h = |a\cos\beta|$. Thus

$$V = |ad\cos\beta| = |\vec{a}\cdot\vec{d}|$$
$$= |\vec{a}\cdot(\vec{b}\times\vec{c})|$$

Let us state (without proof) HW1

$$(*)\qquad \vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b}) \qquad \underline{\text{BAC CAB}}$$

We note that the right side of the above identity is a vector in the plane of $\vec{b}$ and $\vec{c}$. This is to be expected, since the vector $\vec{a}\times(\vec{b}\times\vec{c})$ is $\perp$ to the vector $\vec{b}\times\vec{c}$ which is itself $\perp$ to the plane of $\vec{b}$ and $\vec{c}$.

Let us consider the expression.

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$$

Then by $(*)$ HW2

$$= \vec{c}\,[(\vec{a}\times\vec{b})\cdot\vec{d}\,] - \vec{d}\,[(\vec{a}\times\vec{b})\cdot\vec{c}\,] \qquad (1)$$

Since an interchange of the order of the vectors in a vector product only changes the sign

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = -(\vec{c}\times\vec{d})\times(\vec{a}\times\vec{b})$$

And by $(*)$

$$= -\vec{a}\,[(\vec{c}\times\vec{d})\cdot\vec{b}\,] + \vec{b}\,[(\vec{c}\times\vec{d})\cdot\vec{a}\,] \qquad (2)$$

We next consider the expression

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})$$

If we consider it as a scalar triple product of $(\vec{a}\times\vec{b})$, $\vec{c}$, and $\vec{d}$, and subject these three vectors to two permutations, then according to Thm 1 on p. 1-5

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \vec{c}\cdot[\vec{d}\times(\vec{a}\times\vec{b})].$$

If the vector triple product on the rhs of this equation is expanded by the identity (*), we obtain

$$\begin{aligned}(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) &= (\vec{c}\cdot\vec{a})(\vec{d}\cdot\vec{b}) - (\vec{c}\cdot\vec{b})(\vec{d}\cdot\vec{a}) \\ &= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{d})\end{aligned}$$

## Differentiation of Vectors

A vector $\vec{A}$ may be a function of a scalar quantity, say $t$

$$\vec{A} = \vec{A}(t) = [A_x(t), A_y(t), A_z(t)]$$

$$\frac{d\vec{A}}{dt} = \lim_{\Delta t\to 0}\frac{\vec{A}(t+\Delta t)-\vec{A}(t)}{\Delta t}.$$

We may also define the vector derivative:

$$\frac{d\vec{A}}{dt} = \hat{x}\frac{dA_x}{dt} + \hat{y}\frac{dA_y}{dt} + \hat{z}\frac{dA_z}{dt}$$

The derivatives of vector sums and products obey the rules of ordinary vector calculus.

$$\frac{d}{dt}(\vec{A}+\vec{B}) = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}$$

$$\frac{d}{dt}(\phi\vec{A}) = \frac{d\phi}{dt}\vec{A} + \phi\frac{d\vec{A}}{dt}$$

$$\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \frac{d\vec{A}}{dt}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{dt}$$

$$\frac{d}{dt}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}$$

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Examples of derivatives . . .

Kinematics in a plane.

Let $\vec{r} = \vec{r}(t)$ i.e. $x = x(t)$ & $y = y(t)$

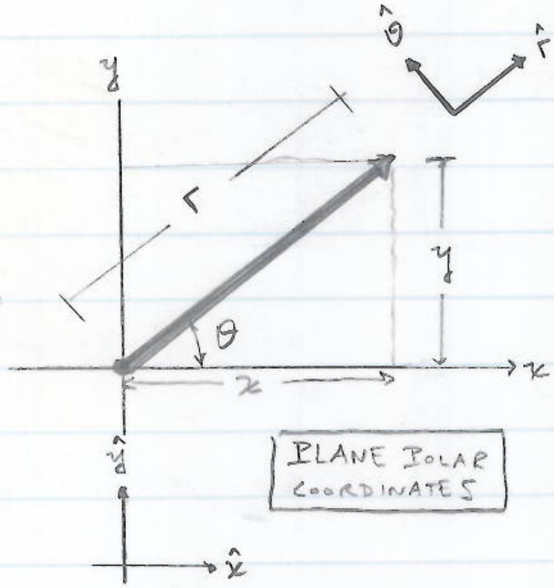
The velocity and acceleration, and their components, are given by

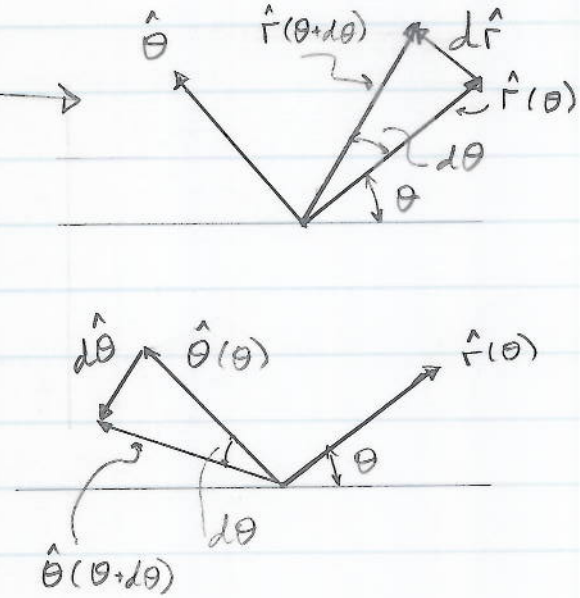
$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{x}\frac{dx}{dt} + \hat{y}\frac{dy}{dt}$$

$$v_x = \frac{dx}{dt} \qquad v_y = \frac{dy}{dt}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt} = \hat{x}\frac{d^2x}{dt^2} + \hat{y}\frac{d^2y}{dt^2}$$

$$a_x = \frac{d^2x}{dt^2} \qquad a_y = \frac{d^2y}{dt^2}$$




INCREMENTS in the vectors $\hat{r}$ & $\hat{\theta}$

The coordinates $r, \theta$ are related to $x$ and $y$ by:

$$x = r\cos\theta \qquad y = r\sin\theta$$

and

$$r = (x^2+y^2)^{1/2}$$

$$\theta = \tan^{-1}\frac{y}{x} = \sin^{-1}\frac{y}{(x^2+y^2)^{1/2}} = \cos^{-1}\frac{x}{(x^2+y^2)^{1/2}}$$

We define the unit vectors $\hat{r}, \hat{\theta}$ in the direction of increasing $r$ and $\theta$, respectively. The vectors $\hat{r}$ and $\hat{\theta}$ are functions of the angle $\theta$, and are related to $\hat{x}$ and $\hat{y}$ by the equations

$$\left.\begin{aligned} \hat{r} &= \hat{x}\cos\theta + \hat{y}\sin\theta \\ \hat{\theta} &= -\hat{x}\sin\theta + \hat{y}\cos\theta \end{aligned}\right\} \text{ These follow from the Fig. above}$$

Note that $\hat{r}\cdot\hat{r} = \hat{\theta}\cdot\hat{\theta} = 1$ ; $\hat{r}\cdot\hat{\theta} = 0$

Differentiating, we obtain the important formulas

$$\frac{d\hat{r}}{d\theta} = \hat{\theta}, \quad \frac{d\hat{\theta}}{d\theta} = -\hat{r}$$

The position vector $\vec{r}$ is given by

$$\vec{r} = r\,\hat{r}(\theta)$$

The velocity vector is given by

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt}$$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

Thus we obtain the components of velocity in the $\hat{r}, \hat{\theta}$ directions: $\quad v_r = \dot{r} \qquad v_\theta = r\dot{\theta}$

The acceleration vector is

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{d\theta}\frac{d\theta}{dt}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$ HW III

The components of acceleration are

$$a_r = \ddot{r} - r\dot{\theta}^2, \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

The term $r\dot{\theta}^2 = \frac{v_\theta^2}{r}$ is called the <u>centripetal acceleration</u> arising from motion in the $\theta$ direction. If $\ddot{r} = \dot{r} = 0$, the path is a circle, and

$$a_r = -\frac{v_\theta^2}{r}$$

This result is familiar from elementary physics.

The term $2\dot{r}\dot{\theta}$ is sometimes called the coriolis acceleration.

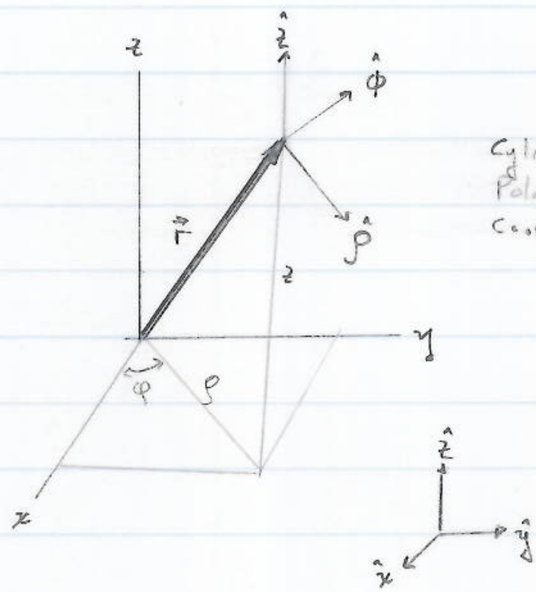
## Kinematics in Three Dimensions

$$\vec{v} = \hat{x}v_x + \hat{y}v_y + \hat{z}v_z$$

&

$$\vec{a} = \hat{x}a_x + \hat{y}a_y + \hat{z}a_z.$$

# Cylindrical Coordinate System




Cylindrical Polar coordinates.

$x = \rho\cos\varphi \quad y = \rho\sin\varphi \quad z = z$

$$\rho = (x^2 + y^2)^{1/2}$$

$$\varphi = \tan^{-1}\frac{y}{x}$$

$$= \sin^{-1}\frac{y}{(x^2+y^2)^{1/2}}$$

$$= \cos^{-1}\frac{x}{(x^2+y^2)^{1/2}}$$

$\hat{\rho}, \hat{\varphi}, \hat{z}$ are in the directions of $\rho, \varphi, z$, respectively

$$\hat{\rho} = \hat{x}\cos\varphi + \hat{y}\sin\varphi \qquad \hat{\varphi} = -\hat{x}\sin\varphi + \hat{y}\cos\varphi$$

$$\frac{d\hat{\rho}}{d\varphi} = \hat{\varphi}, \qquad \frac{d\hat{\varphi}}{d\varphi} = -\hat{\rho}$$

The position vector $\vec{r}$ can be expressed in cylindrical coordinates in the form:

$$\vec{r} = \rho\hat{\rho} + z\hat{z}$$

Differentiating, we obtain for velocity and acceleration

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{\rho}\hat{\rho} + \rho\dot{\varphi}\hat{\varphi} + \dot{z}\hat{z}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = (\ddot{\rho} - \rho\dot{\varphi}^2)\hat{\rho} + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\hat{\varphi} + \ddot{z}\hat{z}$$

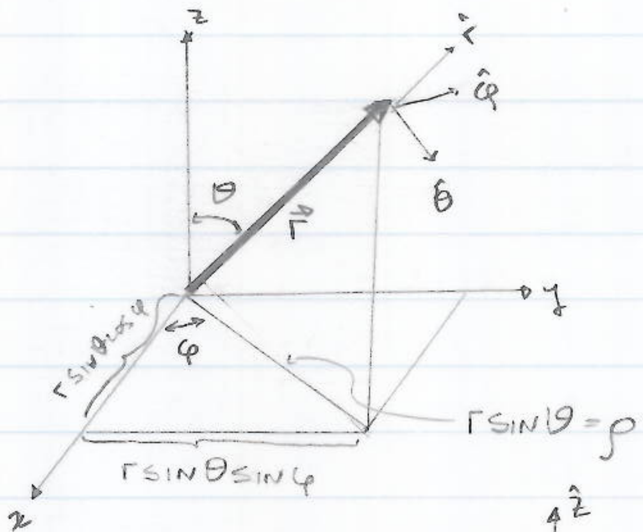
Since $\hat{z}, \hat{\varphi}, \hat{\rho}$ form a set of mutually perpendicular unit vectors, any vector $\vec{A}$ can be expressed in terms of its components along $\hat{z}, \hat{\varphi}, \hat{\rho}$:

$$\vec{A} = A_\rho \hat{\rho} + A_\varphi \hat{\varphi} + A_z \hat{z} \qquad (*)$$

It must be noted that since $\hat{\rho}$ and $\hat{\varphi}$ are functions of $\varphi$, the set of components $(A_\varphi, A_\rho, A_z)$ refers in general to a specific point in space at which the vector $\vec{A}$ is to be located, or at least to a specific value of the coordinate $\varphi$. If $\vec{A}$ is a function of a parameter, say $t$, then we may compute its derivative by differentiating $(*)$. We must, however, be CAREFUL to take into account the variation of $\hat{\rho}$ and $\hat{\varphi}$ if the location of the vector is also changing with $t$ (e.g. if $\vec{A}$ is the force acting on a moving particle):

$$\frac{d\vec{A}}{dt} = \left(\frac{dA_\rho}{dt} - A_\varphi \frac{d\varphi}{dt}\right)\hat{\rho} + \left(\frac{dA_\varphi}{dt} + A_\rho \frac{d\varphi}{dt}\right)\hat{\varphi} + \frac{dA_z}{dt}\hat{z}$$

# Spherical Polar Coordinates

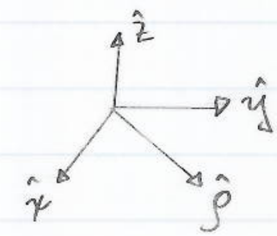



$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

$$r = (x^2+y^2+z^2)^{1/2}$$

$$\theta = \tan^{-1}\frac{(x^2+y^2)^{1/2}}{z}$$

$$\varphi = \tan^{-1}\frac{y}{x}$$




We observe that $\hat{z}$, $\hat{\rho}$, $\hat{r}$, $\hat{\theta}$ all lie in one vertical plane.

The unit vector $\hat{\rho}$ is useful in obtaining relations involving $\hat{r}$ and $\hat{\theta}$

$$\hat{r} = \hat{z}\cos\theta + \hat{\rho}\sin\theta = \hat{z}\cos\theta + \hat{x}\sin\theta\cos\varphi + \hat{y}\sin\theta\sin\varphi$$

$$\hat{\theta} = -\hat{z}\sin\theta + \hat{\rho}\cos\theta = -\hat{z}\sin\theta + \hat{x}\cos\theta\cos\varphi + \hat{y}\cos\theta\sin\varphi$$

$$\hat{\varphi} = -\hat{x}\sin\varphi + \hat{y}\cos\varphi$$

By differentiating the above formulas

$$\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta} \qquad \frac{\partial \hat{r}}{\partial \varphi} = \hat{\varphi}\sin\theta$$

$$\frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r} \qquad \frac{\partial \hat{\theta}}{\partial \varphi} = \hat{\varphi}\cos\theta$$

$$\frac{\partial \hat{\varphi}}{\partial \theta} = 0 \qquad \frac{\partial \hat{\varphi}}{\partial \varphi} = -\hat{\rho} = -\hat{r}\sin\theta - \hat{\theta}\cos\theta$$

In spherical coordinates the position vector is simply

$$\vec{r} = r\hat{r} \leftarrow \text{Note that } \hat{r} = \hat{r}(\theta, \varphi)$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + (r\dot{\varphi}\sin\theta)\hat{\varphi}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta)\hat{\theta}$$
$$+ (r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta)\hat{\varphi}$$

Again, $\hat{r}, \hat{\theta}, \hat{\varphi}$ form a set of mutually perpendicular unit vectors, and any vector $\vec{A}$ may be represented in terms of its spherical components $\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\varphi\hat{\varphi}$

$$\frac{d\vec{A}}{dt} = \left(\frac{dA_r}{dt} - A_\theta\frac{d\theta}{dt} - A_\varphi \sin\theta \frac{d\varphi}{dt}\right)\hat{r}$$
$$+ \left(\frac{dA_\theta}{dt} + A_r\frac{d\theta}{dt} - A_\varphi\cos\theta\frac{d\varphi}{dt}\right)\hat{\theta}$$
$$+ \left(\frac{dA_\varphi}{dt} + A_r\sin\theta\frac{d\varphi}{dt} + A_\theta\cos\theta\frac{d\varphi}{dt}\right)\hat{\varphi}$$

# The Gradient, $\nabla$

Suppose that $\varphi(x, y, z)$ is a scalar point function, that is, a function whose value depends on the values of the coordinates $(x, y, z)$. As a scalar, it must have the same value at a given fixed point in space, <u>independent</u> of the rotation of our coordinate system, or

$$\varphi'(x_1', x_2', x_3') = \varphi(x_1, x_2, x_3)$$

By differentiating wrt $x_i'$ we obtain

$$\frac{\partial \varphi'(x_1', x_2', x_3')}{\partial x_i'} = \frac{\partial \varphi(x_1, x_2, x_3)}{\partial x_i'}$$

$$= \sum_j \frac{\partial \varphi}{\partial x_j} \frac{\partial x_j}{\partial x_i'} = \sum_j a_{ij} \frac{\partial \varphi}{\partial x_j} \qquad (*)$$

Aside

Let us rotate our vector $\vec{r} = x_1 \hat{x}_1 + x_2 \hat{x}_2$ by an angle $\theta$. We have (see p. 1-11)

$$x_1' = a_{11} x_1 + a_{12} x_2 \qquad a_{11} = \cos\theta$$
$$x_2' = a_{21} x_1 + a_{22} x_2 \qquad a_{12} = \sin\theta$$
$$a_{21} = -\sin\theta$$
$$a_{22} = \cos\theta$$

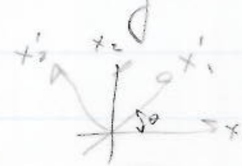
This can be written compactly as

$$x'_i = \sum_{j=1}^{2} a_{ij} x_j \qquad i = 1, 2$$

The generalization to N dimensions is very simple. The set of N quantities, $x_j$, is said to be the components of an N-dimensional vector, $\vec{x}$, if and only if the values in a rotated coordinate system are given by

$$x'_i = \sum_{j=1}^{N} a_{ij} V_j \qquad i = 1, 2, \ldots, N$$

The coefficient $a_{ij}$ may be interpreted as a direction cosine, the cosine of the angle between $x'_i$ and $x_j$



$$a_{12} = \cos(x'_1, x_2) = \sin\theta$$

$$a_{21} = \cos(x'_2, x_1) = \cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta$$

From the definition of $a_{ij}$ as the cosine of the angle between the positive $x'_i$ direction and the positive $x_j$ direction we may write

$$a_{ij} = \frac{\partial x'_i}{\partial x_j} = \frac{\partial x_j}{\partial x'_i}$$

You can prove this differentiating $x'_i = \sum a_{ij} x_k$ wrt $x_j$

Hence

$$x'_i = \sum_{j=1}^{N} \frac{\partial x'_i}{\partial x_j} x_j = \sum_{j=1}^{N} \frac{\partial x_j}{\partial x'_i} x_j$$

The direction cosines $a_{ij}$ satisfy an orthogonality condition

$$\sum_i a_{ij} a_{ik} = \delta_{jk}$$

or equivalently

$$\sum_i a_{ji} a_{ki} = \delta_{jk}$$

The symbol $\delta_{jk}$ is the Kronecker delta defined by

$$\delta_{jk} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

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Back to (*)

$$\frac{\partial \varphi'}{\partial x'_i} = \sum_j a_{ij} \frac{\partial \varphi}{\partial x_j} \qquad a_{ij} : \text{Direction cosines.}$$

By our earlier discussion we have constructed a vector with components $\partial\varphi/\partial x_j$. This vector we call the gradient of $\varphi$

A convenient notation is

$$\nabla\varphi = \hat{x}\frac{\partial\varphi}{\partial x} + \hat{y}\frac{\partial\varphi}{\partial y} + \hat{z}\frac{\partial\varphi}{\partial z}$$

or

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} = \sum_i \hat{x}_i \frac{\partial}{\partial x_i}$$

$\nabla\varphi$ (or del $\varphi$) is our gradient of the scalar $\varphi$, whereas $\nabla$ (del) itself is a vector differential operator. It should be emphasized that this operator is a hybrid creature that must satisfy both the laws for handling vectors and the laws of partial differentiation

<u>Example</u> Calculate the gradient of

$$f(r) = f(\sqrt{x^2+y^2+z^2})$$

$$\nabla f(r) = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$$

Now $f(r)$ depends on $x$ through the dependence of $r$ on $x$. Therefore

$$\frac{\partial f(r)}{\partial x} = \frac{df(r)}{dr}\frac{\partial r}{\partial x} = \frac{df}{dr}\cdot\frac{x}{r}$$

N.B. This is just a special case of the chain rule of partial differentiation

$$\frac{\partial f(r,\theta,\varphi)}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial \varphi}\frac{\partial \varphi}{\partial x}$$

Here $\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial \varphi} = 0$ ; $\frac{\partial f}{\partial r} \rightarrow \frac{df}{dr}$

We have then

$$\nabla f(r) = (\hat{x}x + \hat{y}y + \hat{z}z)\frac{1}{r}\frac{df}{dr}$$

$$= \frac{\vec{r}}{r}\frac{df}{dr} = \hat{r}\frac{df}{dr}$$

One immediate application of $\nabla\varphi$ is to dot it into an increment of length.

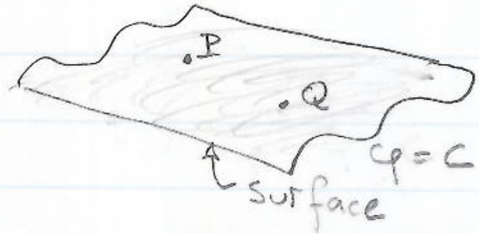
$$d\vec{r} = \hat{x}dx + \hat{y}dy + \hat{z}dz$$

Thus we obtain

$$\nabla\varphi \cdot d\vec{r} = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy + \frac{\partial \varphi}{\partial z}dz$$

$$= d\varphi$$

This is the change in the scalar function $\varphi$ corresponding to a change in position $d\vec{r}$




Now consider P and Q to be two points on a surface $\varphi(x, y, z) = C$, a constant. The points are chosen so that Q is a distance $d\vec{r}$ from P. Then, moving from P to Q, the change in $\varphi(x, y, z) = C$ is given by

$$d\varphi = (\nabla\varphi) \cdot d\vec{r} = 0$$

since we stay on the surface $\varphi(x, y, z) = C$. This shows that $\nabla\varphi$ is perpendicular to $d\vec{r}$. $\nabla\varphi$ is therefore NORMAL to the surface $\varphi = \text{const.}$

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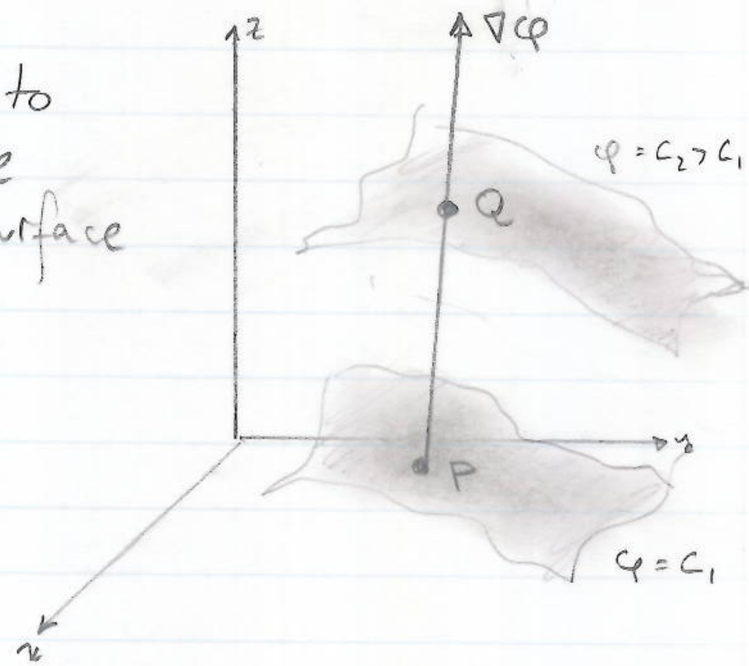


If we now permit $d\vec{r}$ to take us from one surface $\varphi = C_1$ to an adjacent surface $\varphi = C_2$,

$$d\varphi = C_2 - C_1 = \Delta C$$

$$= (\nabla\varphi) \cdot d\vec{r}$$

For a given $d\varphi$, $|d\vec{r}|$ is minimized when it is chosen parallel to $\nabla\varphi$ ($\cos\theta = 1$); or, for a given $|d\vec{r}|$, the change in the scalar function $\varphi$ is maximized by choosing $d\vec{r}$ parallel to $\nabla\varphi$.

$\Rightarrow$ This identifies $\nabla\varphi$ as a vector having the direction of the maximum space & rate of change of $\varphi$.

The gradient of a scalar is of extreme importance in physics in expressing the relation between a force field and a potential field

$$\text{FORCE} = -\nabla(\text{potential})$$

This illustrated by both gravitational and electrostatic fields, among others.

THE LAPLACIAN

The divergence of the gradient, $\nabla\cdot\nabla\varphi$, is named the Laplacian

$$\nabla\cdot\nabla\varphi = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\hat{x}\frac{\partial\varphi}{\partial x} + \hat{y}\frac{\partial\varphi}{\partial y} + \hat{z}\frac{\partial\varphi}{\partial z}\right)$$
$$= \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}.$$