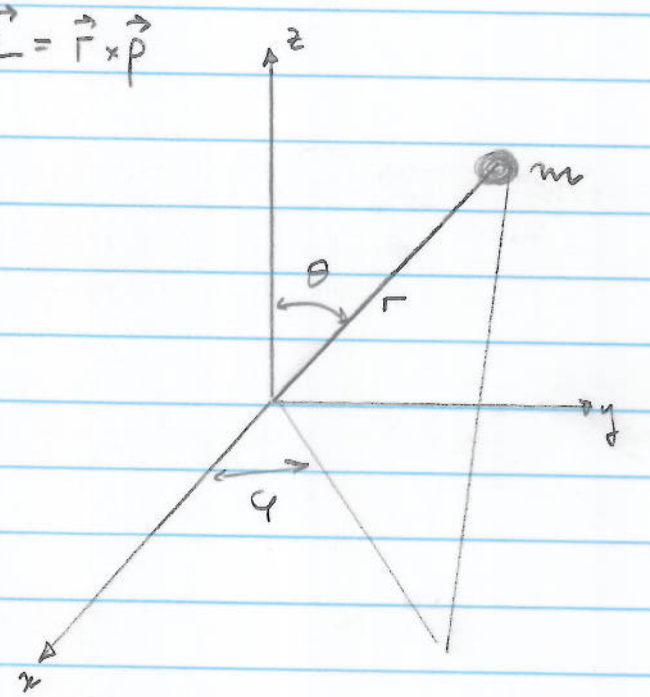
Let us revisit the spherical pendulum in Hamiltonian formulation, using spherical polar coordinates for the $q_i$. We wish to evaluate directly in terms of the canonical variables the following Poisson Brackets:

$$[L_x, L_y]\ ,\quad [L_y, L_z]\ ,\quad [L_z, L_x]$$

$L \equiv$ Angular momentum $\quad \vec{L} = \vec{r} \times \vec{p}$




We found that

$$p_\theta = mr^2\dot{\theta}$$
$$p_\phi = mr^2 \sin^2\theta\, \dot{\varphi}$$

$$L_x = y p_z - z p_y$$
$$= m(y\dot{z} - z\dot{y})$$

$$L_y = z p_x - x p_z$$
$$= m(z\dot{x} - x\dot{z})$$

$$L_z = x p_y - y p_x$$
$$= m(x\dot{y} - y\dot{x})$$

$$x = r\sin\theta\cos\varphi$$
$$\dot{x} = r\dot{\theta}\cos\theta\cos\varphi - r\dot{\varphi}\sin\theta\sin\varphi$$

$$y = r\sin\theta\sin\varphi$$
$$\dot{y} = r\dot{\theta}\cos\theta\sin\varphi + r\dot{\varphi}\sin\theta\cos\varphi$$

$$z = r\cos\theta$$
$$\dot{z} = -r\sin\theta$$

$$L_z = m\Big[(r\sin\theta\cos\varphi)(r\dot\theta\cos\theta\sin\varphi + r\dot\varphi\sin\theta\cos\varphi)$$
$$-(r\sin\theta\sin\varphi)(r\dot\theta\cos\theta\cos\varphi - r\dot\varphi\sin\theta\sin\varphi)\Big]$$

$$= m(r^2\dot\varphi\sin^2\theta) = p_\phi$$ as Determined earlier (see p. 7-49)

$$\boxed{L_z = p_\phi}$$

$$L_x = m\Big[(r\sin\theta\sin\varphi)(-r\dot\theta\sin\theta) - (r\cos\theta)(r\dot\theta\cos\theta\sin\varphi + r\dot\varphi\sin\theta\cos\varphi)\Big]$$
$$= m\Big[-r^2\dot\theta\sin^2\theta\sin\varphi - r^2\dot\theta\cos^2\theta\sin\varphi - r^2\dot\varphi\cos\theta\sin\theta\cos\varphi\Big]$$
$$= -m\Big[r^2\sin\varphi + r^2\dot\varphi\sin^2\theta\cot\theta\cos\varphi\Big]$$

$$\boxed{L_x = -\sin\varphi\, p_\theta - \cot\theta\cos\varphi\, p_\phi}$$

$$L_y = m\Big[(r\cos\theta)(r\dot\theta\cos\theta\cos\varphi - r\dot\varphi\sin\theta\sin\varphi)$$
$$-(r\sin\theta\cos\varphi)(-r\dot\theta\sin\theta)\Big]$$

$$= m\Big[r^2\dot\theta\cos^2\theta\cos\varphi - r^2\dot\varphi\sin\theta\cos\theta\sin\varphi + r^2\dot\theta\sin^2\theta\cos\varphi\Big]$$
$$= m\Big[r^2\dot\theta\cos\varphi - r^2\dot\varphi\sin^2\theta\cot\theta\sin\varphi\Big]$$

$$\boxed{L_y = \cos\varphi\, p_\theta - \cot\theta\sin\varphi\, p_\phi}$$

Now from definition

$$[g, h] = \sum_k \left(\frac{\partial g}{\partial q_k}\frac{\partial h}{\partial p_k} - \frac{\partial g}{\partial p_k}\frac{\partial h}{\partial q_k}\right)$$

7-56

$$[L_x, L_y] = \frac{\partial L_x}{\partial \theta}\frac{\partial L_y}{\partial p_\theta} - \frac{\partial L_x}{\partial p_\theta}\frac{\partial L_y}{\partial \theta} + \frac{\partial L_x}{\partial \varphi}\frac{\partial L_y}{\partial p_\varphi} - \frac{\partial L_x}{\partial p_\varphi}\frac{\partial L_y}{\partial \varphi} ]$$

$$= (+\csc^2\theta \cos\varphi\, p_\varphi)(\cos\varphi) + (\sin\varphi)(+\csc^2\theta \sin\varphi\, p_\varphi)$$
$$+ (-\cos\varphi\, p_\theta + \cot\theta \sin\varphi\, p_\varphi)(-\cot\theta \sin\varphi)$$
$$+ (\cot\theta\cos\varphi)(-\sin\varphi\, p_\theta - \cot\theta\cos\varphi\, p_\varphi)$$

$$= \frac{\cos^2\varphi}{\sin^2\theta} p_\varphi + \frac{\sin^2\varphi}{\sin^2\theta} p_\varphi + \cot\theta\cos\varphi\sin\varphi\, p_\theta$$

$$- \cot^2\theta\sin^2\varphi\, p_\varphi - \cot\theta\cos\varphi\sin\varphi\, p_\theta - \cot^2\theta\cos^2\varphi\, p_\varphi$$

$$= (+\csc^2\theta - \cot^2\theta)\, p_\phi = p_\phi = L_z$$

$$\boxed{[L_x, L_y] = L_z}$$

$$[L_y, L_z] = \frac{\partial L_y}{\partial \theta}\frac{\partial L_z}{\partial p_\theta} - \frac{\partial L_y}{\partial p_\theta}\frac{\partial L_z}{\partial \theta} + \frac{\partial L_y}{\partial \varphi}\frac{\partial L_z}{\partial p_\phi} - \frac{\partial L_y}{\partial p_\phi}\frac{\partial L_z}{\partial \phi}$$

$$= (\csc^2\theta\sin\varphi\, p_\phi)(0) - 0 + (-\sin\varphi\, p_\theta)(1)$$
$$- (\cot\theta\cos\varphi\, p_\phi)(1)$$

$$= -(\sin\varphi\, p_\theta + \cot\theta\cos\varphi\, p_\phi) = L_x$$

$$\boxed{[L_y, L_z] = L_x}$$

$$[L_z, L_x] = -\frac{\partial L_z}{\partial p_\phi}\frac{\partial L_x}{\partial \varphi}$$ ; all other terms equal zero

$$= (-1)(-\cos\varphi\, p_\theta + \cot\theta \sin\varphi\, p_\varphi)$$

$$= \cos\varphi\, p_\theta - \cot\theta\sin\varphi\, p_\varphi$$

$$\boxed{[L_z, L_x] = L_y}$$

Poisson's Thm  The Poisson Bracket of any two constants of the motion is ALSO a constant of the motion

Thus, if any two components of the angular momentum are constant, the TOTAL angular momentum vector is conserved.

That is, for example if $L_x = \text{const}$ and $L_y = \text{const}$, then $L_z$ is also a constant of the motion.

As a further instance, let us assume that in addition to $L_x$ and $L_y$ being conserved there is a cartesian vector of canonical momentum $\vec{p}$ with $p_z$ a constant of the motion. Not only then is $L_z$ conserved but we have two further constants of the motion:
$$[p_z, L_x] = p_y$$
$$[p_z, L_y] = -p_x$$
That is both $\vec{L}$ and $\vec{p}$ are conserved!