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→ Advice. In your careers as professional physicists you will need to make extensive use of integral tables. You should buy one....

Momentum & Energy Theorems

Let us first study one-dimensional motion.....

$$m\frac{d^2x}{dt^2} = F$$

The motion of a particle is governed by Newton's Second Law.

The linear momentum p is defined:

$$p = mv_x = m\frac{dx}{dt}$$

Hence

$$\frac{dp}{dt} = F$$

$$p_2 - p_1 = \int_{t_1}^{t_2} F\,dt$$

This is the impulse delivered by a force F during the time interval of $t_2 - t_1$

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A quantity of considerable importance in classical mechanics is the kinetic energy defined:

$$T = \frac{1}{2} m v^2$$

Now $\quad m \dfrac{dv}{dt} = F$

$$m v \frac{dv}{dt} = F v$$

$$= \frac{d}{dt}\left(\frac{1}{2} m v^2\right) = \frac{dT}{dt} = F v$$

$$\Rightarrow T_2 - T_1 = \int_{t_1}^{t_2} F v \, dt$$

Since $v = \dfrac{dx}{dt}$ we can rewrite

$$T_2 - T_1 = \int_{x_0}^{x} F \, dx$$

! $\Longrightarrow$ The change in kinetic energy equals the work performed over the interval $x - x_0$ by force F.

(Work - Energy Thm)

We now define the potential energy $U(x)$ as the work done by the force when the particle goes from x to some standard point x_s

$$U(x) = \int_x^{x_s} F(x)\,dx = -\int_{x_s}^{x} F(x)\,dx$$

This serves as the reference point

$$\int_{x_0}^{x} F(x)\,dx = -U(x) + U_0$$

or

$$\frac{1}{2}mv^2 + U(x) = \frac{1}{2}mv_0^2 + U(x_0)$$

The quantity on the right depends only on the initial conditions and is therefore a constant of the motion. It is called the TOTAL ENERGY E

$$\frac{1}{2}mv^2 + U(x) = T + U = E$$

This conservation law holds ONLY when the force is a function of position alone

* This is the CONSERVATION OF MECHANICAL ENERGY

Nonconservative Forces

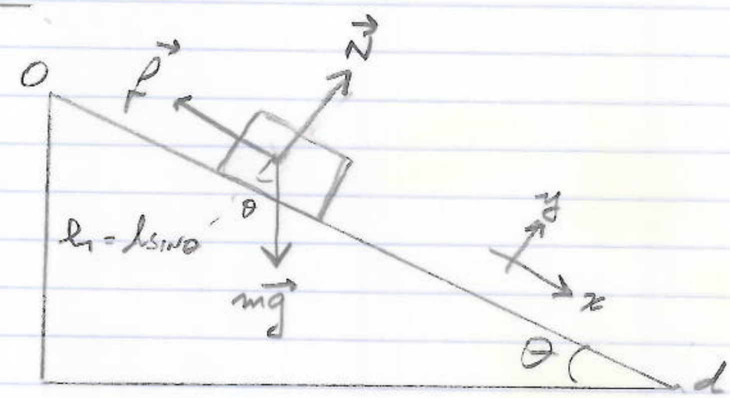
The Work-Energy Thm: $\Delta K = W = W_{con} + \underbrace{W_{nc}}_{\text{nonconservative}}$

$\Delta E \equiv \Delta(K+U) = W_{nc}$ (since for conservative forces $\Delta E = 0$ or Mechanical Energy is conserved)

This is to say that the Mechanical Energy changes exactly as much as the work done on that body.

Block Sliding down incline

$\vec{N} + \vec{f} + m\vec{g} = m\vec{a}$ NII



We can also solve this through Energy techniques

$U = mgh = mgd\sin\theta$

Note that the Normal Force does no work $\vec{N} \perp$ Direction of travel, $\hat{x}$

We observe $\boxed{\Delta K + \Delta U = W_{nc}}$

$\Delta K = \frac{1}{2}mv^2$ $\Delta U = mgd\sin\theta$ $W_{nc} = \vec{f}_{Fric} \cdot d\hat{x} = \mu mg\cos\theta\, d$ ($\mu mg\cos\theta\,\hat{x}$)

Hence

$$\frac{1}{2}mv^2 - mgd\sin\theta = -\mu mgd\cos\theta$$

$$v = \sqrt{2gd(\sin\theta - \mu\cos\theta)}$$

Q: If I were to endow the brick with a speed of $v_0 = v$ above at the bottom of the incline, how far up would it go?

$T_i = \frac{1}{2}\frac{2gd(\sin\theta - \mu\cos\theta)}{m}$ $u_i = 0$ $u_f = mgh'$ $T_f = 0$

$T_i = U_f \Rightarrow h' = d(\sin\theta - \mu\cos\theta)$

(1) $W(\vec{r} \to \vec{r} + d\vec{r}) = F(\vec{r}) \cdot d\vec{r}$

$= F_x dx + F_y dy + F_z dz$

$W(\vec{r} \to \vec{r} + d\vec{r}) = -dU = -[U(\vec{r} + d\vec{r}) - U(\vec{r})]$

$= -[U(x+dx, y+dy, z+dz) - U(x,y,z)]$

Now (for x direction) $f = f(x)$

$df = f(x+dx) - f(x) = \frac{df}{dx} dx$ ← Definition of the derivative

So for $U = U(x,y,z)$

$$dU = \underbrace{\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz}_{\text{(PARTIAL Derivatives)}}$$

(2) $W(\vec{r} \to \vec{r} + d\vec{r}) = -\left[\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz\right]$

comparing (1) and (2)

$$F_x = -\frac{\partial U}{\partial x} \qquad F_y = -\frac{\partial U}{\partial y} \qquad F_z = -\frac{\partial U}{\partial z}$$

$$\vec{F} = -\hat{x}\frac{\partial U}{\partial x} - \hat{y}\frac{\partial U}{\partial y} - \hat{z}\frac{\partial U}{\partial z}$$

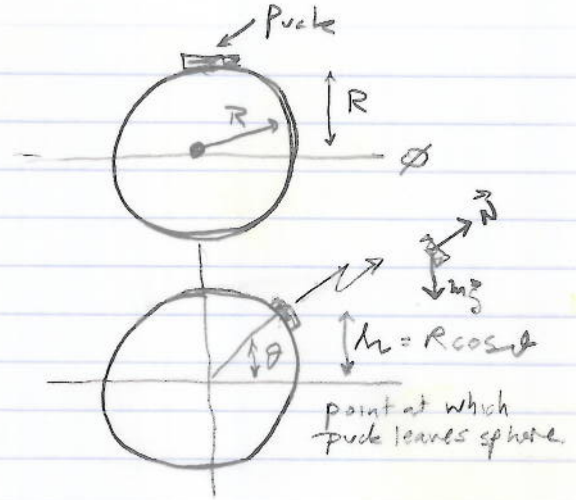
We observe that $\nabla f = \underbrace{\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)}_{\text{operator}} f = \text{grad} f$

∇ = del, nabla, grad

$\boxed{\vec{F} = -\nabla U}$ ← The Force $\vec{F}$ is derivable from a potential energy

↑ True iff $U = U(x,y,z)$ $\quad U \neq U(\dot{x}, \dot{y}, \dot{z})$!

8. By the conservation of Mechanical Energy



$$E_i = E_f$$

$$\cancelto{0}{T_i} + U_i = T_f + U_f$$

$$mgR = \frac{1}{2}mv^2 + mgR\cos\theta$$

$$m\frac{v^2}{R} = 2mg(1-\cos\theta) \qquad (1)$$

Now the radial component of NII is $N - mg\cos\theta = -m\frac{v^2}{R}$ (2)

The Puck leave the surface at $N=0$

$$(2') = m\frac{v^2}{R} = mg\cos\theta$$

Equating (1) and (2')

$$\cancel{mg}\cos\theta = 2\cancel{mg}(1-\cos\theta)$$

$$3\cos\theta = 2 \Rightarrow \theta = \arccos\left(\frac{2}{3}\right) \qquad \boxed{\theta = 48^\circ}$$

Ex

Given the two-dimensional potential function

$$V(\vec r) = V_0 - \frac{1}{2}k\delta^2 e^{-r^2/\delta^2}$$

Find the Force function, $\vec F = x\hat i + y\hat j$ $\quad \left.\begin{matrix} V_0 \\ k \\ \delta \end{matrix}\right\}$ constants

Rewriting

$$V(x,y) = V_0 - \frac{1}{2}k\delta^2 e^{-(x^2+y^2)/\delta^2}$$

$$\vec F = -\nabla V = -(\hat i x + \hat j y)e^{-(x^2+y^2)/\delta^2}$$

$$\boxed{\vec F = -k\vec r e^{-r^2/\delta^2}}$$

Ex

Suppose a particle of mass m is moving in the above force field, and at $t=0$ the particle passes through the origin with speed v_0. What will be the speed of the particle at some distance away from the origin given by $\vec{r} = (\Delta r)\hat{r}$ $\Delta r \ll \delta$

Because the potential energy function exists, the ENERGY IS CONSTANT

conserved

$$E = \frac{1}{2}mv^2 + V(\vec{r}) = \frac{1}{2}mv_0^2 + V(0)$$

Solving for v

$$v^2 = v_0^2 + \frac{2}{m}\left[V(0) - V(\vec{r})\right]$$

$$= v_0^2 + \frac{2}{m}\left[\left(V_0 - \frac{1}{2}k\delta^2\right) - \left(V_0 - \frac{1}{2}k\delta^2 e^{-(\Delta r)^2/\delta^2}\right)\right]$$

$$= v_0^2 - \frac{k\delta^2}{m}\left[1 - e^{-(\Delta r)^2/\delta^2}\right]$$

$\downarrow$ small $\Delta r \Rightarrow$ Taylor Expansion

$\leftarrow$ FIRST TWO terms

$$\cong v_0^2 - \frac{k\delta^2}{m}\left[1 - \left(1 - \frac{(\Delta r)^2}{\delta^2}\right)\right]$$

$$\boxed{v^2 \cong v_0^2 - \frac{k}{m}(\Delta r)^2}$$

We may rewrite the above equation as:

$$V(t) = \frac{dx}{dt} = \pm\sqrt{\frac{2}{m}[E - U(x)]}$$

And by integrating:

$$t - t_0 = \int_{x_0}^{x_0} \frac{\pm dx}{\sqrt{\frac{2}{m}[E - U(x)]}} \qquad (*)$$

In applying this relation, and in taking the indicated square root in the integrand, care must be taken to use the proper sign, depending on whether the velocity V is positive or negative. In cases where V is positive during some parts of the motion and negative during other parts, it may become necessary to carry out the integration SEPARATELY for each part of the motion

From the definition $-U(x) = \int F dx$ we can express the force in terms of the potential energy:

$$F = -\frac{dU}{dx}$$

This equation can be taken as the PHYSICAL MEANING of the Potential Energy.

As an example, we consider the problem of a particle subject to a linear restoring force, for example, a mass fastened to a spring:

$$F = -kx$$

The potential energy, if we take $x_s = 0$, is

$$U(x) = -\int_0^x (-kx)\,dx$$

$$= \frac{1}{2}kx^2$$

Equation (*) becomes, for this case, with $t_0 = 0$

$$t = \sqrt{\frac{m}{2}} \int_{x_0}^{x} \left(E - \frac{1}{2}kx^2\right)^{-1/2} dx$$

Now we make the substitutions:

$$\sin\theta = x\sqrt{\frac{k}{2E}} \qquad \omega = \sqrt{\frac{k}{m}}$$

so that t now can be expressed as

$$t = \frac{1}{\omega}\int_{\theta_0}^{\theta} d\theta = \frac{1}{\omega}(\theta - \theta_0)$$

$$\Rightarrow \quad \theta = \omega t + \theta_0$$

We can now solve for x

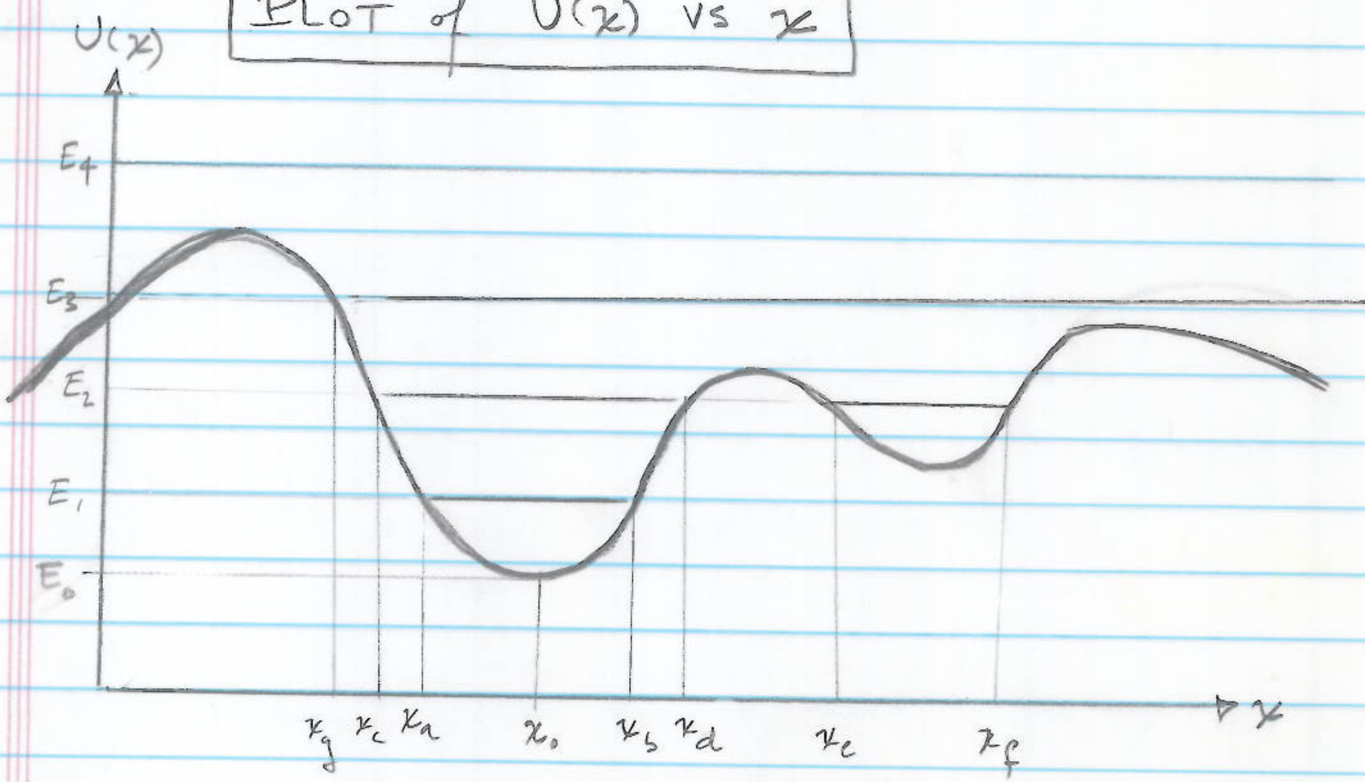
$$x = \sqrt{\frac{2E}{k}} \sin\theta = A \sin(\omega t + \theta_0)$$

where $A = \sqrt{\frac{2E}{k}}$

Thus the coordinate x oscillates harmonically in time, with amplitude A and frequency $\omega/2\pi$. The initial conditions are here determined by the constants A and θ_0, which are related to E and x_0 by

$$E = \frac{1}{2} k A^2 \quad \text{and} \quad x_0 = A \sin\theta_0$$

$\boxed{\text{PLOT of } U(x) \text{ vs } x}$



First observe that because $T = \frac{1}{2}mv^2 \geq 0$, $E \geq U(x)$ for any physical motion. We see in the above figure that the motion is <u>BOUNDED</u> for energies E_1 and E_2. For E_1, the motion is <u>periodic</u> between <u>turning points</u> x_a and x_b. Similarly, for E_2 the motion is periodic, but there are two possible regions: $x_c \leq x \leq x_d$, and $x_e \leq x \leq x_f$. The particle cannot "jump" from one "pocket" to the other; once in a pocket, it must remain there forever if its energy remains at E_2. The motion for a particle with energy E_0 has only one value, $x = x_0$. The particle is at rest with $T = 0$ $[E_0 = U(x_0)]$

Local Minimum

The motion for a particle with energy E_3 is simple: The particle comes in from infinity, stops, and turns at $x = x_g$, and returns to infinity — much like a tennis ball bouncing against a wall. For energy E_4, the motion is unbounded and the particle may be at any position. Its speed will change because it depends on the difference between E_4 and $U(x)$; that is in accordance with the depth of the potential at each point

When a particle is oscillating near a point of stable equilibrium, i.e between points x_a and x_b, we can find an approximate solution for its motion. Let $U(x)$ have a minimum at $x = x_0$, and expand the function $U(x)$ in a Taylor series about this point

$$U(x) = U(x_0) + \left(\frac{dU}{dx}\right)_{x_0}(x - x_0) + \frac{1}{2!}\left(\frac{d^2U(x)}{dx^2}\right)_{x_0}(x - x_0)^2 + \frac{1}{3!}\left(\frac{d^3U(x)}{dx^3}\right)_{x_0}(x - x_0)^3 + \ldots$$

We evaluate this at $x = x_0$.

The potential energy at $U(x_0)$ is simply a constant which we can set to ZERO without affecting the physical results. Since x_0 is a minimum point

$$\left(\frac{dU}{dx}\right)_{x_0} = 0 \; ; \quad \left(\frac{d^2U}{dx^2}\right)_{x = x_0} > 0$$

EQUILIBRIUM POINT

CONCAVE UP
[Stable equilibrium]

2-23.

Making the abbleviations:

$$k = \left(\frac{d^2U}{dx^2}\right)_{x=x_0} \quad ; \quad x' = x - x_0$$

we can write the potential energy function as.

$$U(x') = \frac{1}{2}kx'^2 + \dots$$

For sufficiently small values of x', provided $k' \neq 0$, we may neglect the higher order terms.

Because x'^2 is always positive, the conditions for the equilibrium are:

$$\left(\frac{d^2U}{dx^2}\right)_{x_0} > 0 \quad \text{Stable Equilibrium}$$

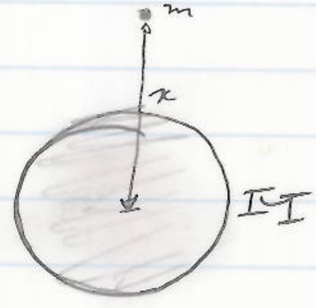
$$\left(\frac{d^2U}{dx^2}\right)_{x_0} < 0 \quad \text{Unstable Equilibrium.}$$

if $\left(\frac{d^2U}{dx^2}\right)_{x_0}$ is zero, then higher-order terms must be examined.

Let us recapitulate: A point where $U(x)$ has a minimum is called a point of stable equilibrium. A particle at rest at such a point will remain at rest. If displaced a slight distance, it will experience a restoring force tending to return it, and it will oscillate about the point of equilibrium. A point where $U(x)$ has a maximum is called a point of UNSTABLE EQUILIBRIUM. In theory, a particle at rest there can remain at rest, since the force is zero, but if it displaced the slightest distance, the force acting on it will push it farther away from the unstable equilibrium position

Example

In the case of bodies falling from a great height, the variation of the gravitational force with height should be taken into account. In this case, we neglect air resistance and measure x from the center of the earth. Then if M is the mass of the earth and m the mass of the falling body



$$F = -G\frac{Mm}{x^2}$$

and

$$U(x) = -\int_{\infty}^{x} F\,dx = -G\frac{Mm}{x}$$

where we have taken $x_s = \infty$ (in order to avoid the constant term in $U(x)$

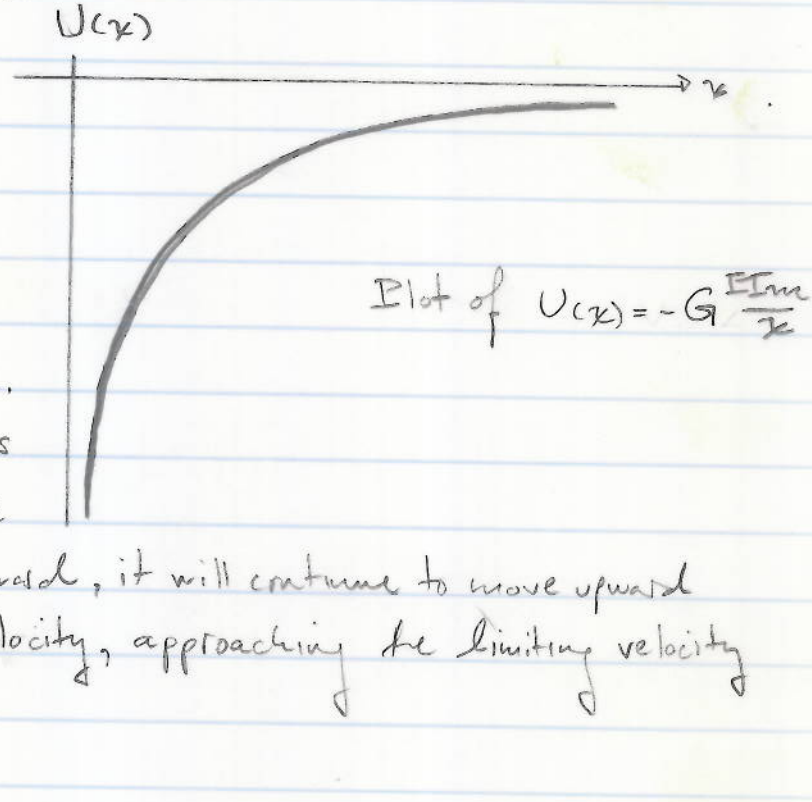
We showed on p. 2-18

$$v = \frac{dx}{dt} = \pm\sqrt{\frac{2}{m}}\,(E - U(x))^{1/2}$$

which becomes.

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}}\left(E + G\frac{\mathrm{II}m}{x}\right)^{1/2}$$

The plus sign refers to ascending motion, the minus sign to descending motion.



We see that there are two types of motion, depending on whether E is positive or negative. When E is positive, there is no turning point, and if the body is initially moving upward, it will continue to move upward forever, with decreasing velocity, approaching the limiting velocity

$$V_l = \sqrt{\frac{2E}{m}}$$

When E is negative, there is a turning point at a height

$$x_T = G\frac{m\mathrm{II}}{(-E)}$$

If the body is initially moving upward, it will come to a stop at x_T and will fall back to earth.

The dividing case between these types of motion occurs when the initial position and velocity are such that $E=0$. The turning point is then at infinity, and the body moves upward forever, approaching the limiting value of $v_l = 0$. If $E=0$, then at any height x, the velocity will be

$$v_e = \sqrt{\frac{2GM}{x}}$$

This is called the escape velocity for a body at a distance x from the center of the earth.

To find $x(t)$ we must evaluate the integral

$$\int_{x_0}^{x} \frac{dx}{\pm\left(E + G\frac{mM}{x}\right)} = \sqrt{\frac{2}{m}}\, t \qquad (*)$$

where x_0 is the height at $t=0$. To solve for the case when $E<0$, we substitute

$$\cos u = \Bigg[\frac{-Ex}{\underbrace{GmM}_{x_T}}\Bigg]^{1/2}$$

(*) becomes:

$$\frac{GmM}{(-E)^{3/2}}\int_{u_0}^{u} 2\cos^2 u'\, du' = \sqrt{\frac{2}{m}}\, t$$

Without loss of generality, we can take x_0 to be at the turning point x_T, since the body will at some time in its past or future pass through x_T if no force except gravity acts upon it. Provided $E<0$ then $u_0 = 0$, and

$$\frac{Gm\Sigma}{(-E)^{3/2}}(u + \sin u \cos u) = \sqrt{\frac{2}{m}}\, t$$

or

(**) $$u + \frac{1}{2}\sin 2u = \left[\frac{2\Sigma G}{x_T^3}\right]^{1/2} t$$

and

$$x = x_T \cos^2 u$$

for determining $x = x(t)$

These pair of equations cannot be solved explicitly for $x(t)$. A numerical solution can be obtained by choosing a sequence of values of u and finding the corresponding values of x and t