Dylan J. Temples: Moller Scattering

Particle Physics
Elementary Particle Physics in a Nutshell - M. Tully
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1 Moller Scattering.

Compute the differential cross section for $e^-e^- \to e^-e^-$.

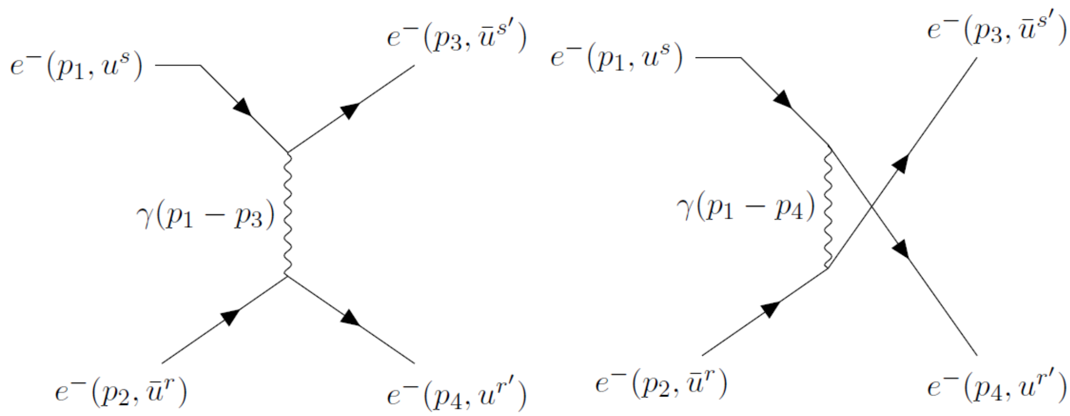

Figure 1: The tree-level Feynman diagrams describing the process $e^+e^- \to e^+e^-$. (Left) t-channel. (Right) u-channel.

The tree-level diagrams for the process $e^-e^- \to e^-e^-$ are shown in figure 1, the left diagram representing the t-channel and the right diagram representing the u-channel.

1.1 t-channel.

From the diagram, we can write down the matrix element:

$$\mathcal{M}_t = (e\bar{u}_3\gamma^\mu u_1)\frac{ig_{\mu\nu}}{q^2}(e\bar{u}_4\gamma_\nu u_2) = \frac{e^2}{q^2}[\bar{u}_3\gamma_\mu u_1][\bar{u}_4\gamma^\mu u_2] \ , \tag{1}$$

where the propagator momentum $q = p_1 - p_3 = \sqrt{t}$. The modulus squared is

$$|\mathcal{M}_t|^2 = \frac{e^4}{t^2}[\bar{u}_3\gamma_\mu u_1][\bar{u}_4\gamma^\mu u_2][\bar{u}_3\gamma_\nu u_1]^\dagger[\bar{u}_4\gamma^\nu u_2]^\dagger \tag{2}$$

$$= \frac{e^4}{t^2}[\bar{u}_3\gamma_\mu u_1][\bar{u}_4\gamma^\mu u_2][\bar{u}_1\gamma_\nu u_3][\bar{u}_2\gamma^\nu u_4] \tag{3}$$

$$= \frac{e^4}{t^2}[\bar{u}_3\gamma_\mu u_1\bar{u}_1\gamma_\nu u_3][\bar{u}_2\gamma^\nu u_4\bar{u}_4\gamma^\mu u_2] \ , \tag{4}$$

where we've used the identity:

$$[\bar{u}\gamma^\nu v]^\dagger = [u^\dagger\gamma^0\gamma^\nu v]^\dagger = v^\dagger\gamma^{\nu\dagger}\gamma^{0\dagger}u = v^\dagger(\gamma^0)^2\gamma^{\nu\dagger}\gamma^0 u = \bar{v}(\gamma^0\gamma^{\nu\dagger}\gamma^0)u = \bar{v}\gamma^\nu u \ , \tag{5}$$

which holds for u and v spinors. Since the quantities in the square brackets are C-numbers, we can take their traces with no repercussions:

$$|\bar{\mathcal{M}_t}|^2 = \frac{e^4}{t^2}\,\mathrm{Tr}[\bar{u}_3\gamma_\mu u_1\bar{u}_1\gamma_\nu u_3]\,\mathrm{Tr}[\bar{u}_2\gamma^\nu u_4\bar{u}_4\gamma^\mu u_2] = \frac{e^4}{t^2}\,\mathrm{Tr}[u_1\bar{u}_1\gamma_\nu u_3\bar{u}_3\gamma_\mu]\,\mathrm{Tr}[u_2\bar{u}_2\gamma^\nu u_4\bar{u}_4\gamma^\mu] \ . \quad (6)$$

Now we must average over the initial spins and sum over the final spins:

$$|\bar{\mathcal{M}_t}|^2 = \frac{e^4}{4t^2}\,\mathrm{Tr}\left[\sum_s u_1^s\bar{u}_1^s\gamma_\nu \sum_{s'} u_3^{s'}\bar{u}_3^{s'}\gamma_\mu\right]\mathrm{Tr}\left[\sum_r u_2^r\bar{u}_2^r\gamma^\nu \sum_{r'} u_4^{r'}\bar{u}_4^{r'}\gamma^\mu\right] \ , \quad (7)$$

we use the completeness relations for Dirac spinors:

$$|\bar{\mathcal{M}_t}|^2 = \frac{e^4}{4t^2}\,\mathrm{Tr}\left[(\not{p}_1 - m_e)\gamma_\nu(\not{p}_3 - m_e)\gamma_\mu\right]\mathrm{Tr}\left[(\not{p}_2 - m_e)\gamma^\nu(\not{p}_4 - m_e)\gamma^\mu\right] \ , \quad (8)$$

here we will note that any term which is linear in the electron mass will contain an odd number of Dirac matrices inside a trace, which evaluates to zero. Therefore:

$$|\bar{\mathcal{M}_t}|^2 = \frac{e^4}{4t^2}\left\{\mathrm{Tr}\left[\not{p}_1\gamma_\nu\not{p}_3\gamma_\mu\right] + m_e^2\,\mathrm{Tr}\left[\gamma_\nu\gamma_\mu\right]\right\}\left\{\mathrm{Tr}\left[\not{p}_2\gamma^\nu\not{p}_4\gamma^\mu\right] + m_e^2\,\mathrm{Tr}\left[\gamma^\nu\gamma^\mu\right]\right\} \quad (9)$$

$$= \frac{e^4}{4t^2}\left\{p_1^\alpha p_3^\beta\,\mathrm{Tr}\left[\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\mu\right] + m_e^2\,\mathrm{Tr}\left[\gamma_\nu\gamma_\mu\right]\right\}\left\{p_{2\alpha}p_{4\beta}\,\mathrm{Tr}\left[\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\mu\right] + m_e^2\,\mathrm{Tr}\left[\gamma^\nu\gamma^\mu\right]\right\} \ . \quad (10)$$

Using the trace identities of Dirac matrices (Peskin Appendix 3) this is

$$|\bar{\mathcal{M}_t}|^2 = \frac{e^4}{4t^2}\left\{p_1^\alpha p_3^\beta\,\mathrm{Tr}\left[\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\mu\right] + 4g_{\mu\nu}m_e^2\right\}\left\{p_{2\alpha}p_{4\beta}\,\mathrm{Tr}\left[\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\mu\right] + 4g^{\mu\nu}m_e^2\right\} \ , \quad (11)$$

the remaining traces are

$$p_1^\alpha p_3^\beta\,\mathrm{Tr}\left[\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\mu\right] = 4p_1^\alpha p_3^\beta\left(g_{\alpha\nu}g_{\beta\mu} - g_{\alpha\beta}g_{\nu\mu} + g_{\alpha\mu}g_{\nu\beta}\right) \quad (12)$$

$$= 4\left(p_{1\nu}p_{3\mu} - (p_1\cdot p_3)g_{\nu\mu} + p_{1\mu}p_{3\nu}\right) \quad (13)$$

$$p_{2\alpha}p_{4\beta}\,\mathrm{Tr}\left[\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\mu\right] = p_{2\alpha}p_{4\beta}\,\mathrm{Tr}\left[\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\mu\right] = 4p_{2\alpha}p_{4\beta}\left(g^{\alpha\nu}g^{\beta\mu} - g^{\alpha\beta}g^{\nu\mu} + g^{\alpha\mu}g^{\nu\beta}\right) \quad (14)$$

$$= 4\left(p_2^\nu p_4^\mu - (p_2\cdot p_4)g^{\nu\mu} + p_2^\mu p_4^\nu\right) \ . \quad (15)$$

Now we must do the multiplication in the matrix element. The first term is the contraction of the above two traces, so using notation (1+2+3)(A+B+C), the terms are

$$\text{1A} : \ (p_2\cdot p_1)(p_4\cdot p_3) \quad (16)$$

$$\text{1B} : \ -(p_1\cdot p_3)(p_2\cdot p_4) \quad (17)$$

$$\text{1C} : \ (p_4\cdot p_1)(p_2\cdot p_3) \quad (18)$$

$$\text{4A} : \ -(p_2\cdot p_4)(p_1\cdot p_3) \quad (19)$$

$$\text{4B} : \ 4(p_2\cdot p_4)(p_1\cdot p_3) \quad (20)$$

$$\text{4C} : \ -(p_2\cdot p_4)(p_1\cdot p_3) \quad (21)$$

$$\text{3A} : \ (p_2\cdot p_3)(p_4\cdot p_1) \quad (22)$$

$$\text{3B} : \ -(p_1\cdot p_3)(p_2\cdot p_4) \quad (23)$$

$$\text{3C} : \ (p_2\cdot p_1)(p_4\cdot p_3) \ , \quad (24)$$

which we must sum (including an overall factor of 4^2), note the factor of 4 in 2B comes from $g^{\mu\nu}g_{\mu\nu} = 4$. Let us note the relations:

$$1\text{A} = 3\text{C} \qquad 1\text{C} = 3\text{A} \qquad 2\text{B} + 1\text{B} + 2\text{A} + 2\text{C} + 3\text{B} = 0 \ , \tag{25}$$

so their sum is

$$4^2 \left[2(p_2 \cdot p_1)(p_4 \cdot p_3) + 2(p_4 \cdot p_1)(p_2 \cdot p_3)\right] \ . \tag{26}$$

The second term (in the matrix element) is the contraction of the two mass terms:

$$4^2 g_{\mu\nu} g^{\mu\nu} m_e^4 = 4^2 \left[4m_e^4\right] \ . \tag{27}$$

The remaining two terms are the "cross-terms":

$$4\left(p_{1\nu}p_{3\mu} - (p_1 \cdot p_3)g_{\nu\mu} + p_{1\mu}p_{3\nu}\right) 4g^{\mu\nu}m_e^2 = 4^2 \left[m_e^2\left(p_1 \cdot p_3 - 4p_1 \cdot p_3 + p_1 \cdot p_3\right)\right] \tag{28}$$

$$= 4^2 \left[-2m_e^2 p_1 \cdot p_3\right] \ , \tag{29}$$

and

$$4\left(p_2^\nu p_4^\mu - (p_2 \cdot p_4)g^{\nu\mu} + p_2^\mu p_4^\nu\right) 4g_{\mu\nu}m_e^2 = 4^2 \left[m_e^2\left(p_2 \cdot p_4 - 4(p_2 \cdot p_4) + p_2 \cdot p_4\right)\right] \tag{30}$$

$$= 4^2 \left[-2m_e^2 p_2 \cdot p_4\right] \ . \tag{31}$$

Gathering these terms, the matrix element is

$$|\bar{\mathcal{M}}_t|^2 = 4\frac{e^4}{t^2}\left\{2(p_2 \cdot p_1)(p_4 \cdot p_3) + 2(p_4 \cdot p_1)(p_2 \cdot p_3) + 4m_e^4 - 2m_e^2(p_1 \cdot p_3) - 2m_e^2(p_2 \cdot p_4)\right\} \ . \tag{32}$$

Again we will use our Mandelstam invariants, noting (in the massive case)

$$2p_1 \cdot p_2 = s - 2m_e^2 = 2p_3 \cdot p_4 \tag{33}$$

$$-2p_1 \cdot p_3 = t - 2m_e^2 = -2p_2 \cdot p_4 \tag{34}$$

$$-2p_1 \cdot p_4 = u - 2m_e^2 = -2p_2 \cdot p_3 \ . \tag{35}$$

With these, the matrix element squared is

$$\begin{aligned}
|\bar{\mathcal{M}}_t|^2 &= 2\frac{e^4}{t^2}\left\{2(p_1 \cdot p_2)2(p_4 \cdot p_3) + (-2)(p_4 \cdot p_1)(-2)(p_2 \cdot p_3) + 8m_e^4 - 4m_e^2(p_1 \cdot p_3) - 4m_e^2(p_2 \cdot p_4)\right\} \\
&= 2\frac{e^4}{t^2}\left\{(s - m_e^2)^2 + (u - m_e^2)^2 + 8m_e^4 + 2m_e^2(t - 2m_e^2) + 2m_e^2(t - 2m_e^2)\right\} \\
&= 2\frac{e^4}{t^2}\left\{(s - m_e^2)^2 + (u - m_e^2)^2 + 8m_e^4 + 4m_e^2(t - 2m_e^2)\right\} \\
&= 2\frac{e^4}{t^2}\left\{s^2 + m_e^4 - 2sm_e^2 + u^2 + m_e^4 - 2um_e^2 + 8m_e^4 + 4m_e^2 t - 8m_e^4\right\} \\
|\bar{\mathcal{M}}_t|^2 &= 2\frac{e^4}{t^2}\left\{s^2 + u^2 + 2m_e^2(2t - s - u) + 2m_e^4\right\} \ .
\end{aligned}$$

We recover, in the ultra-relativistic limit (*i.e.*, massless electron):

$$|\bar{\mathcal{M}}_t|^2 = 2e^4\frac{s^2 + u^2}{t^2} \ . \tag{36}$$

1.2 u-channel.

From the diagram, we can write down the matrix element:

$$\mathcal{M}_u = (e\bar{u}_4\gamma^\mu u_1)\frac{ig_{\mu\nu}}{q^2}(e\bar{u}_3\gamma_\nu u_2) = \frac{e^2}{q^2}[\bar{u}_4\gamma_\mu u_1][\bar{u}_3\gamma^\mu u_2] \ , \tag{37}$$

the analysis that follows will be abbreviated because the process is nearly identical as the t-channel. The modulus squared of the matrix element is

$$|\mathcal{M}_u|^2 = \left(\frac{e^2}{q^2}\right)^2 [\bar{u}_4\gamma_\mu u_1][\bar{u}_3\gamma^\mu u_2][\bar{u}_1\gamma_\nu u_4][\bar{u}_2\gamma^\nu u_3] \tag{38}$$

$$= \frac{e^4}{u^2}[\bar{u}_4\gamma_\mu u_1][\bar{u}_1\gamma_\nu u_4][\bar{u}_2\gamma^\nu u_3][\bar{u}_3\gamma^\mu u_2] \tag{39}$$

$$= \frac{e^4}{u^2}\,\mathrm{Tr}[u_1\bar{u}_1\gamma_\nu u_4\bar{u}_4\gamma_\mu]\,\mathrm{Tr}[u_2\bar{u}_2\gamma^\nu u_3\bar{u}_3\gamma^\mu] \ , \tag{40}$$

now we sum and average over final and initial spins, and use the completeness relations for the Dirac spinors:

$$|\bar{\mathcal{M}}_u|^2 = \frac{e^4}{4u^2}\,\mathrm{Tr}[(\not{p}_1 + m)\gamma_\nu(\not{p}_4 + m)\gamma_\mu]\,\mathrm{Tr}[(\not{p}_2 + m)\gamma^\nu(\not{p}_3 + m)\gamma^\mu] \tag{41}$$

$$= \frac{e^4}{4u^2}\left\{\mathrm{Tr}\left[\not{p}_1\gamma_\nu\not{p}_4\gamma_\mu\right] + m_e^2\,\mathrm{Tr}\left[\gamma_\nu\gamma_\mu\right]\right\}\left\{\mathrm{Tr}\left[\not{p}_2\gamma^\nu\not{p}_3\gamma^\mu\right] + m_e^2\,\mathrm{Tr}\left[\gamma^\nu\gamma^\mu\right]\right\} \tag{42}$$

$$= \frac{e^4}{4u^2}\left\{p_1^\alpha p_4^\beta\,\mathrm{Tr}\left[\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\mu\right] + 4g_{\mu\nu}m_e^2\right\}\left\{p_{2\alpha}p_{3\beta}\,\mathrm{Tr}\left[\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\mu\right] + 4g^{\mu\nu}m_e^2\right\} \tag{43}$$

the remaining traces are

$$p_1^\alpha p_4^\beta\,\mathrm{Tr}[\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\mu] = 4p_1^\alpha p_4^\beta\left(g_{\alpha\nu}g_{\beta\mu} - g_{\alpha\beta}g_{\nu\mu} + g_{\alpha\mu}g_{\nu\beta}\right) \tag{44}$$

$$= 4\left(p_{1\nu}p_{4\mu} - (p_1\cdot p_4)g_{\nu\mu} + p_{1\mu}p_{4\nu}\right) \tag{45}$$

$$p_{2\alpha}p_{3\beta}\,\mathrm{Tr}[\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\mu] = 4p_{2\alpha}p_{3\beta}\left(g^{\alpha\nu}g^{\beta\mu} - g^{\alpha\beta}g^{\nu\mu} + g^{\alpha\mu}g^{\nu\beta}\right) \tag{46}$$

$$= 4\left(p_2^\nu p_3^\mu - (p_2\cdot p_3)g^{\nu\mu} + p_2^\mu p_3^\nu\right) \ , \tag{47}$$

which contract to give

$$4^2\left[2(p_2\cdot p_1)(p_3\cdot p_4) + 2(p_3\cdot p_1)(p_2\cdot p_4)\right] \ . \tag{48}$$

The product of the mass terms is

$$4^2\left[4m_e^4\right] \ , \tag{49}$$

and the cross-terms are

$$4\left(p_{1\nu}p_{4\mu} - (p_1\cdot p_4)g_{\nu\mu} + p_{1\mu}p_{4\nu}\right)4m_e^2 g^{\mu\nu} = 4^2\left[-2m_e^2(p_1\cdot p_4)\right] \tag{50}$$

$$4\left(p_2^\nu p_3^\mu - (p_2\cdot p_3)g^{\nu\mu} + p_2^\mu p_3^\nu\right)4m_e^2 g_{\mu\nu} = 4^2\left[-2m_e^2(p_2\cdot p_3)\right] \ , \tag{51}$$

so the matrix element squared is

$$|\bar{\mathcal{M}}_u|^2 = 4\frac{e^4}{u^2}\left\{2(p_2\cdot p_1)(p_3\cdot p_4) + 2(p_3\cdot p_1)(p_2\cdot p_4) + 4m_e^4 - 2m_e^2(p_1\cdot p_4) - 2m_e^2(p_2\cdot p_3)\right\} \ . \tag{52}$$

Writing this in terms of Mandelstam invariants, we have

$$\begin{aligned}
|\bar{\mathcal{M}}_u|^2 &= 2\frac{e^4}{u^2}\left\{2(p_2\cdot p_1)2(p_3\cdot p_4)+(-2)(p_3\cdot p_1)(-2)(p_2\cdot p_4)+8m_e^4-4m_e^2(p_1\cdot p_4)-4m_e^2(p_2\cdot p_3)\right\}\\
&= 2\frac{e^4}{u^2}\left\{(s-m_e^2)^2+(t-m_e^2)^2+8m_e^4+2m_e^2(u-m_e^2)+2m_e^2(u-m_e^2)\right\}\\
|\bar{\mathcal{M}}_u|^2 &= 2\frac{e^4}{u^2}\left\{s^2+t^2+2m_e^2(2u-s-t)+4m_e^4\right\}\ .
\end{aligned}$$

We recover, in the ultra-relativistic limit (*i.e.*, massless electron):

$$|\bar{\mathcal{M}}_u|^2 = 2e^4\frac{s^2+t^2}{u^2}\ . \tag{53}$$

It is interesting to note we could have found this result if we had made the substitutions $p_3 \to p_4$ and $p_4 \to p_3$ (note we do not need to consider a change of sign, because we are not dealing with any antiparticles), so the Mandelstam invariants become

$$s = (p_1+p_2)^2 \to (p_1+p_2)^2 = s \tag{54}$$

$$t = (p_3-p_1)^2 \to (p_4-p_1)^2 = u \tag{55}$$

$$u = (p_4-p_1)^2 \to (p_3-p_1)^2 = t\ . \tag{56}$$

1.3 Cross Terms.

The first cross-term is

$$\bar{\mathcal{M}}_t\bar{\mathcal{M}}_u^* = \frac{1}{4}\sum_{\text{spins}}\left[\frac{e^2}{t}[\bar{u}_3\gamma_\mu u_1][\bar{u}_4\gamma^\mu u_2]\right]\left[\frac{e^2}{u^2}[\bar{u}_4\gamma_\nu u_1]^\dagger[\bar{u}_3\gamma^\nu u_2]^\dagger\right] \tag{57}$$

$$= \frac{e^4}{4tu}\sum_{\text{spins}}[\bar{u}_3\gamma_\mu u_1][\bar{u}_1\gamma_\nu u_4][\bar{u}_4\gamma^\mu u_2][\bar{u}_2\gamma^\nu u_3] \tag{58}$$

$$= \frac{e^4}{4tu}\sum_{\text{spins}}\text{Tr}[\gamma_\mu u_1\bar{u}_1\gamma_\nu u_4\bar{u}_4\gamma^\mu u_2\bar{u}_2\gamma^\nu u_3\bar{u}_3] \tag{59}$$

$$= \frac{e^4}{4tu}\text{Tr}[\gamma_\mu(\not{p}_1+m)\gamma_\nu(\not{p}_4+m)\gamma^\mu(\not{p}_2+m)\gamma^\nu(\not{p}_3+m)] \tag{60}$$

$$= \frac{e^4}{4tu}g_{\mu\alpha}g_{\nu\beta}\,\text{Tr}[\gamma^\alpha(\not{p}_1+m)\gamma^\beta(\not{p}_4+m)\gamma^\mu(\not{p}_2+m)\gamma^\nu(\not{p}_3+m)] \tag{61}$$

$$= \frac{e^4}{4st}g_{\mu\alpha}g_{\nu\beta}\,\text{Tr}[\not{p}_1\gamma^\beta\not{p}_4\gamma^\mu\not{p}_2\gamma^\nu\not{p}_3\gamma^\alpha]\ , \tag{62}$$

For the sake of calculations, let's cast this in a different form, using $p_1 \to p$, $p_2 \to k$, $p_3 \to p'$, and $p_4 \to k'$, we can then write the cross term as

$$\bar{\mathcal{M}}_t\bar{\mathcal{M}}_u^* = \frac{e^4}{4st}g_{\mu\alpha}g_{\nu\beta}\,\text{Tr}[\not{p}\gamma^\beta\not{k}'\gamma^\mu\not{k}\gamma^\nu\not{p}'\gamma^\alpha] = \frac{e^4}{4st}g_{\mu\alpha}g_{\nu\beta}\,\text{Tr}[\not{p}'\gamma^\alpha\not{p}\gamma^\beta\not{k}'\gamma^\mu\not{k}\gamma^\nu] \tag{63}$$

now relabel our indeces (swapping β and μ):

$$\bar{\mathcal{M}}_t\bar{\mathcal{M}}_u^* = \frac{e^4}{4tu}g_{\alpha\beta}g_{\mu\nu}\,\text{Tr}[\not{p}'\gamma^\alpha\not{p}\gamma^\mu\not{k}'\gamma^\beta\not{k}\gamma^\nu]\ . \tag{64}$$

Note the following relations which can be found in Appendix A.4 of Schwartz *Quantum Field Theory and the Standard Model*:

$$\gamma^\mu\gamma_\mu = 4 \tag{65}$$

$$\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu \tag{66}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho} \tag{67}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu\ , \tag{68}$$

and note the fourth holds for any odd number of gamma matrices between two contracted gamma matrices. Therefore:

$$\bar{\mathcal{M}}_t\bar{\mathcal{M}}_u^* = \frac{e^4}{4tu}g_{\alpha\beta}\,\text{Tr}[\not{p}'\gamma^\alpha\not{p}\gamma^\mu\not{k}'\gamma^\beta\not{k}\gamma_\mu] = -\frac{e^4}{2tu}g_{\alpha\beta}\,\text{Tr}[\not{p}'\gamma^\alpha\not{p}\not{k}'\gamma^\beta\not{k}] \tag{69}$$

$$= -\frac{e^4}{2tu}\text{Tr}[\not{p}'\gamma^\alpha\not{p}\not{k}\gamma_\alpha\not{k}'] = -2\frac{e^4}{tu}g^{\rho\sigma}p_\rho k_\sigma\,\text{Tr}[\not{p}'\not{k}'] \tag{70}$$

$$= -2\frac{e^4}{tu}(p\cdot k)\,\text{Tr}[\not{p}'\not{k}'] = -8\frac{e^4}{tu}(p\cdot k)(p'\cdot k')\ . \tag{71}$$

Now in terms of the originally defined momenta:

$$\bar{\mathcal{M}}_t\bar{\mathcal{M}}_u^* = -8\frac{e^4}{tu}(p_1\cdot p_2)(p_3\cdot p_4)\ , \tag{72}$$

in terms of the Mandelstam invariants this is

$$\bar{\mathcal{M}}_t \bar{\mathcal{M}}_u^* = -8\frac{e^4}{tu}\left(\frac{s}{2}\right)\left(\frac{s}{2}\right) = -2e^4\frac{s^2}{tu} \ . \tag{73}$$

Instead of calculating the other cross-term, I will claim

$$2\Re\{\bar{\mathcal{M}}_t \bar{\mathcal{M}}_u^*\} = \bar{\mathcal{M}}_t \bar{\mathcal{M}}_u^* + \bar{\mathcal{M}}_t^* \bar{\mathcal{M}}_u = 2\Re\{\bar{\mathcal{M}}_t^* \bar{\mathcal{M}}_u\} \ , \tag{74}$$

so

$$\bar{\mathcal{M}}_t \bar{\mathcal{M}}_u^* + \bar{\mathcal{M}}_t^* \bar{\mathcal{M}}_u = -4e^4\frac{s^2}{tu} \ . \tag{75}$$

1.4 Differential cross-section.

For a two-body final state, the differential cross-section is given by (Peskin eq. 5.12)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2E_{cm}^2}\frac{|\mathbf{k}|}{16\pi^2 E_{cm}}|\bar{\mathcal{M}}|^2 \ . \tag{76}$$

In the massless limit, center of mass frame, the differential cross-section simplifies to

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2E_{cm}^2}\frac{E_{cm}/2}{16\pi^2 E_{cm}}|\bar{\mathcal{M}}|^2 = \frac{1}{4E_{cm}^2}\frac{1}{(4\pi)^2}|\bar{\mathcal{M}}|^2 \ . \tag{77}$$

Using the matrix elements from the massless limit sections, we have

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_t = \frac{1}{4E_{cm}^2}\frac{1}{(4\pi)^2}2\frac{e^4}{t^2}\left\{s^2+u^2\right\} = \left(\frac{e^2}{4\pi}\right)^2\frac{1}{2s}\frac{s^2+u^2}{t^2} = \frac{\alpha^2}{2s}\frac{s^2+u^2}{t^2} \tag{78}$$

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_u = \frac{1}{4E_{cm}^2}\frac{1}{(4\pi)^2}2\frac{e^4}{u^2}\left\{s^2+t^2\right\} = \left(\frac{e^2}{4\pi}\right)^2\frac{1}{2s}\frac{s^2+t^2}{u^2} = \frac{\alpha^2}{2s}\frac{s^2+t^2}{u^2} \tag{79}$$

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{ut} = \frac{1}{4E_{cm}^2}\frac{1}{(4\pi)^2}(-4)\frac{e^4}{tu}s^2 = -\left(\frac{e^2}{4\pi}\right)^2\frac{1}{s}\frac{s^2}{tu} = -\frac{\alpha^2}{s}\frac{s^2}{tu} \ . \tag{80}$$

Now we need to calculate the sum of the differential cross-sections, but we need first determine their sign. Since there are identical fermions in the final state, we have that

$$\mathcal{M} = \mathcal{M}_t - \mathcal{M}_u \quad \Rightarrow \quad |\mathcal{M}|^2 = |\mathcal{M}_t|^2 + |\mathcal{M}_t|^2 - \bar{\mathcal{M}}_t\bar{\mathcal{M}}_u^* - \bar{\mathcal{M}}_t^*\bar{\mathcal{M}}_u \tag{81}$$

since under particle interchange, fermionic systems are odd. Then

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_t + \left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_u - \left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{ut} , \tag{82}$$

putting in our results:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{2s}\left[\frac{s^2+u^2}{t^2} + \frac{s^2+t^2}{u^2} - (-2)\frac{s^2}{tu}\right] = \frac{\alpha}{2s}\left[\left(\frac{u}{t}\right)^2 + \left(\frac{t}{u}\right)^2 + s^2\left(\frac{1}{u^2}+\frac{1}{t^2}+\frac{2}{tu}\right)\right] , \tag{83}$$

which can be written

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{2s}\left[\left(\frac{u}{t}\right)^2 + \left(\frac{t}{u}\right)^2 + s^2\left(\frac{1}{t}+\frac{1}{u}\right)^2\right] \ . \tag{84}$$

If we integrate over the azimuth, we acquire a factor of 2π, yielding

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{\pi\alpha^2}{s}\left[\left(\frac{u}{t}\right)^2 + \left(\frac{t}{u}\right)^2 + s^2\left(\frac{1}{t}+\frac{1}{u}\right)^2\right] \ . \tag{85}$$

Let's investigate the angular dependence of the differential cross section. In the center of mass frame, we have

$$p_1 = (E_{cm}/2, \mathbf{p}) \qquad p_3 = (E_{cm}/2, \mathbf{k}) \tag{86}$$

$$p_2 = (E_{cm}/2, -\mathbf{p}) \qquad p_4 = (E_{cm}/2, -\mathbf{k}) \ , \tag{87}$$

where $|\mathbf{p}| = |\mathbf{k}| = E_{cm}/2$. Using this, the Mandelstam invariants can be written

$$s = 2p_1 \cdot p_2 = E_{cm}^2 \tag{88}$$
$$t = -2p_1 \cdot p_3 = -2(p_1^0 p_3^0 - (\mathbf{p}\cdot\mathbf{k})) \tag{89}$$
$$= -2\frac{E_{cm}^2}{4}(1-\cos\theta) \tag{90}$$
$$u = -2p_1 \cdot p_4 = -2(p_1^0 p_4^0 - (\mathbf{p}\cdot -\mathbf{k})) \tag{91}$$
$$= -2\frac{E_{cm}^2}{4}(1+\cos\theta) \ . \tag{92}$$

Inserting these into the differential-cross section yields

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{\pi\alpha^2}{s}\Bigg[\left(\frac{-2\frac{E_{cm}^2}{4}(1+\cos\theta)}{-2\frac{E_{cm}^2}{4}(1-\cos\theta)}\right)^2 + \left(\frac{-2\frac{E_{cm}^2}{4}(1-\cos\theta)}{-2\frac{E_{cm}^2}{4}(1+\cos\theta)}\right)^2 + E_{cm}^4\left(\frac{1}{-2\frac{E_{cm}^2}{4}(1-\cos\theta)} + \frac{1}{-2\frac{E_{cm}^2}{4}(1+\cos\theta)}\right)^2\Bigg] \ , \tag{93}$$

simplification yields

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{\pi\alpha^2}{s}\Bigg[\left(\frac{1+\cos\theta}{1-\cos\theta}\right)^2 + \left(\frac{1-\cos\theta}{1+\cos\theta}\right)^2 + 4\left(\frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta}\right)^2\Bigg] \ , \tag{94}$$

letting our old pal Mathematica handle the algebra, we obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x} = \frac{\pi\alpha^2}{E_{cm}^2}\left\{\frac{2\left(x^2+3\right)^2}{\left(x^2-1\right)^2}\right\} \ , \tag{95}$$

where $x = \cos\theta$. We can integrate the angular dependence out, to see the dependence on center of mass energy:

$$\sigma = \int \frac{\pi\alpha^2}{E_{cm}^2}\left\{\frac{2\left(x^2+3\right)^2}{\left(x^2-1\right)^2}\right\}\mathrm{d}x = 2\frac{\pi\alpha^2}{E_{cm}^2}\left\{x - \frac{8x}{x^2-1}\right\} \ , \tag{96}$$

which diverges at ± 1. If we restrict measurement for particles with a scattering angle greater than 10°, we (Mathematica) can evaluate the integral numerically:

$$\tilde{\sigma} = \int_{\cos(170\frac{\pi}{180})}^{\cos(10\frac{\pi}{180})} \frac{\pi\alpha^2}{E_{cm}^2}\left\{\frac{2\left(x^2+3\right)^2}{\left(x^2-1\right)^2}\right\}\mathrm{d}x = 1049.05\frac{\pi\alpha^2}{E_{cm}^2} \ , \tag{97}$$

note the units are GeV^{-2}, so we pick up a factor:

$$\tilde{\sigma} = 1049.05\frac{\pi\alpha^2}{E_{cm}^2}\left(\frac{0.3894 \text{ mb}}{1 \text{ GeV}^{-2}}\right) = \frac{0.06837}{(E_{cm}/\mathrm{GeV})^2} \text{ mb} \ . \tag{98}$$

We now need to include a factor of 1/2 due to the identical particles in the final state[1], so

$$\tilde{\sigma} = \frac{0.0342}{(E_{cm}/\mathrm{GeV})^2} \text{ mb} \ . \tag{99}$$

For example, in colliding electron beams, each with momentum 4000 GeV (so a center-of-mass energy of 8000 GeV), the total cross section (for scattering larger than 10°) is $\sigma = 5.342 \times 10^{-10}$ millibarn, or 0.534 pb.

[1] In short, a detector can't tell the difference between which electron it detects, so to avoid double counting we need to include a factor of 1/2 in the phase space factor. See Peskin pages 107-108, in general we include a factor of $1/n!$ for a final state with n identical particles.

2 CalcHep Comparison.

We now move to the computational tool CalcHep to verify our result for the process `ee→ee` in the Standard Model option, excluding diagrams propagated with a Z-boson. In our model we want to set the electron charge to $e = \sqrt{4\pi\alpha} \simeq \sqrt{4\pi/137} = 0.302862$. After starting `n_calchep`, the `IN state` was set to give each particle a momentum of 4000 GeV. In order to get convergent results, we apply a cut on `A(e)` with `Min bound = 10` and `Max bound = 170`. The `Monte Carlo simulation` resulted in a value of $\sigma = 0.534$ pb, which is in excellent agreement with our calculated result.

A second simulation was run, this time with a cut on the cosine of the scattering angle: $-0.95 \leq x \leq 0.95$. The total cross-section calculated using equation 99 is 0.1607 pb. Again, in excellent agreement with the simulation: $\sigma = 0.1607$ pb. This time the angular distribution (equation 95) of the electron is plotted with respect to $x = \cos\theta$, shown in figure 2 on linear and logarithmic scales. We see excellent agreement between the formula and simulation for the angular distribution (the agreement becomes worse at large angles). For the angular distribution, wee did not need to include the factor of 2 for identical particles, because the $1/n!$ factor only enters when one integrates over the full phase-space. Thus it has no effect on the differential cross-section.

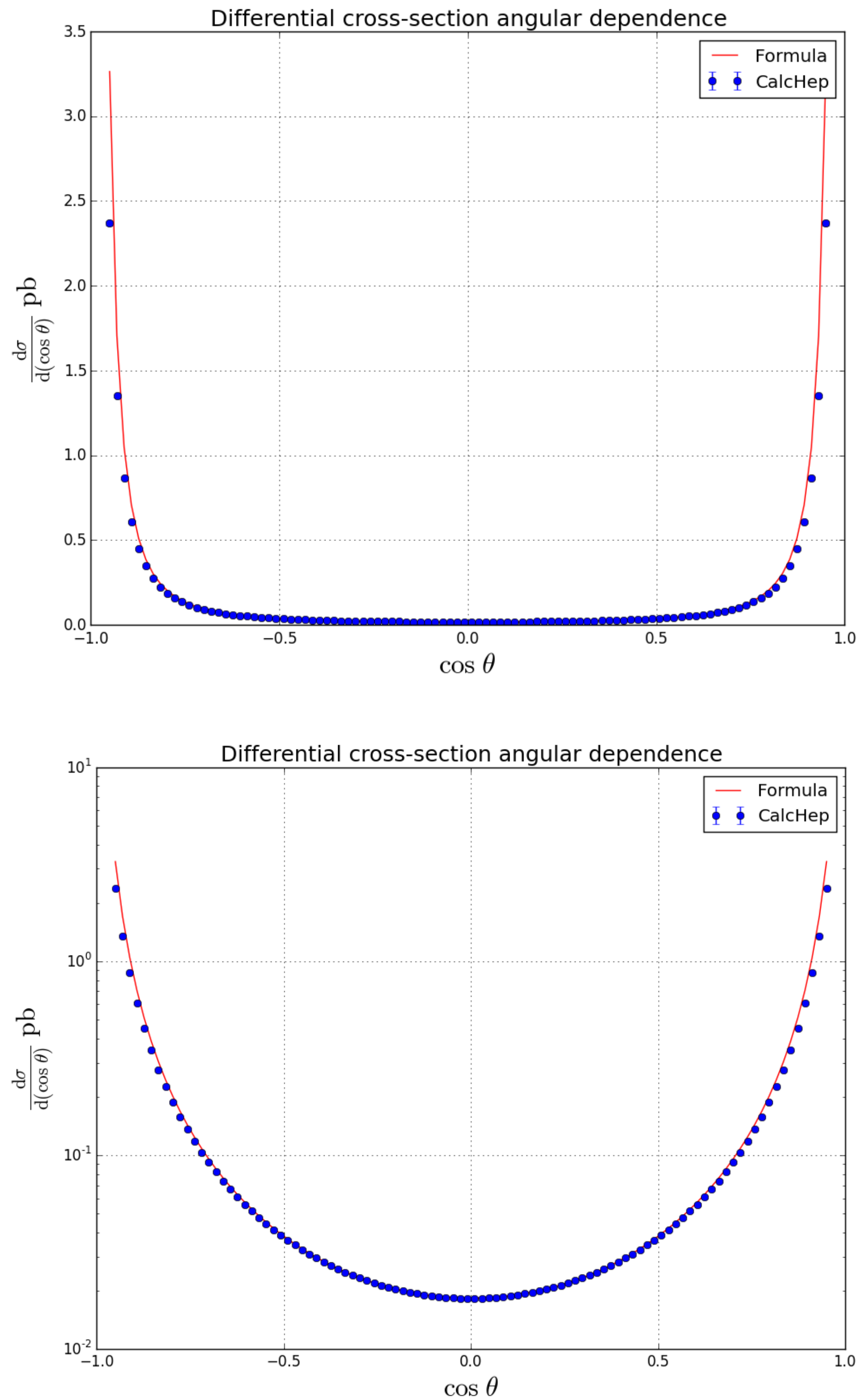

Figure 2: Angular distribution of one of the final state electrons, comparison on CalcHep simulation and analytic calculation. Shown on both linear and logarithmic scales.