Proton Bremsstrahlung*

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The cross section for proton bremsstrahlung is calculated from Sommerfeld's work and expressed as the classical orbit result plus a first quantum correction. This correction is due to the energy loss of the proton to radiation. For 150-kev x-rays emitted at 90° to a beam of 2-Mev protons incident on a tin ($Z=50$) target, this correction decreases the classical prediction by 25 percent and serves to establish theoretical agreement with experimental results of Mark, McClelland, and Goodman.

I. INTRODUCTION

THE bremsstrahlung emitted when low-energy protons are scattered by nuclei has been studied in recent experiments by Zupančič and Huus[1] and by Mark, McClelland, and Goodman.[2] The latter observe x-rays in a 3-kev energy interval about 150 kev which are emitted at 90° to a beam of 2-Mev protons incident on a tin ($Z=50$) target.[2] As reported by Mark, the differential cross section for this process is

$$\sigma(90°)=(1.3\pm0.5)\times10^{-31}\ \text{cm}^2/\text{sterad}. \tag{1}$$

We present here a theoretical analysis of the above result. Since the proton radiates only 7.5 percent of its energy and has a deBroglie wavelength, $\lambda\!\!\!^{-}_D=3.3\times10^{-13}$ cm, which is quite small on an atomic scale, we turn to a classical orbit picture of the process as a convenient starting point. We expect that the effects of retardation, electronic shielding, and finite nuclear size can be neglected, since the x-ray wavelength, $\lambda\!\!\!^{-}=1.3\times10^{-10}$ cm, is 36 times greater than the distance of closest approach (r_m) of the proton to the tin nucleus ($r_m=3.6\times10^{-12}$ cm); and r_m is 28 times smaller than the K-shell Bohr orbit for tin, $a_z=a_0/Z=10^{-10}$ cm, and six times larger than the nuclear radius $R_N=6\times10^{-13}$ cm.

We consider then first of all the dipole radiation emitted classically by a beam of particles scattered in a Coulomb field as discussed, for example, by Landau and Lifschitz.[3] The cross section derived on this basis in the following section (II) is found to be

$$\sigma_{\text{classical}}(90°)=2.1\times10^{-31}\ \text{cm}^2/\text{sterad}, \tag{2}$$

in disagreement with the experimental result, Eq. (1).

We are thus led to perform a more complete quantum-mechanical calculation, taking into account the not inconsiderable energy loss of the proton to radiation. Sommerfeld and his co-workers have investigated at length the quantum-mechanical problem of bremsstrahlung from charged particles moving in Coulomb fields.[4] In particular, Sommerfeld has derived the differential cross section for dipole bremsstrahlung emission in a given direction, with the particle recoiling at fixed angle (Fig. 1). Integration of this result over all proton recoil angles was possible only in special limiting cases which do not apply here (e.g., hard x-rays, which slow down the particle considerably). Sommerfeld also achieved the integration over both x-rays and recoil angles for the total cross section.

The contribution of the present paper is a deduction from Sommerfeld's work of the cross section for dipole x-ray emission as a function of x-ray direction and energy and of incident particle energy only. Our work is valid for parameter choices such as discussed above which permit an expansion in powers of $(\hbar\omega/E)\ll1$ about the classical result ($\hbar\omega$ is the x-ray energy, and E the proton energy). Essentially then we reduce the bremsstrahlung cross section to the classical result plus the first-order quantum correction. The quantum correction arises from the decrease in proton energy after it radiates. This damping effect of the bremsstrahlung on the motion of the proton is not included in the classical orbit calculation. We expect it to decrease the cross section from the classical value of Eq. (2) since the proton is slowed down and repelled further away from the Coulomb field source as it loses energy. The effective dipole moment is thereby reduced. Quantum mechanically, there is a decrease in wave function overlap. We calculate this decrease in Sec. III and show that it suffices to establish agreement with experiment. It is also of interest in this calculation to exhibit explicitly a parameter in terms of which to expand the cross section about its classical value. The Coulomb field (in the absence of any shielding length) has the well-known property of giving rise to elastic scattering cross sections which do not involve $\hbar$ and which are essentially classical in nature. This point has been analyzed in great detail by Williams[5] and is a consequence of the large effective impact parameters in Coulomb scattering. However, the quantum corrections to the classical orbit treatment of bremsstrahlung arise from the damping of the orbit and can be ex-

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[1] Č. Zupančič and T. Huus, Phys. Rev. **94**, 205 (1954).
[2] Mark, McClelland, and Goodman (private communication to be published).
[3] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Cambridge, 1951), Chap. 9.
[4] A. Sommerfeld, *Wellenmechanik* (Frederick Ungar, New York, 1950), Chap. 7.
[5] E. J. Williams, Revs. Modern Phys. **17**, 217 (1945).

pressed in the form $(\hbar\omega/E)$, the measure of the inelasticity.

II. CLASSICAL CALCULATION

We outline in this section the classical calculation of the dipole radiation from a beam of protons scattered by nuclei of charge Z and mass number A. In particular, we investigate the intensity and angular distribution of the radiation emitted by the scattered protons.

First, we briefly consider the scattering of a single proton by such a nucleus. At a large distance R_0 from the radiating system, in the wave zone of the radiation, the ωth Fourier component of the vector potential of the radiation field is given (through $l=2$ multipoles) by

$$\mathbf{A}_\omega = e^{ikR_0+i\mathbf{k}\cdot\mathbf{R}_0} \times\left[\frac{d\mathbf{d}_\omega/dt}{cR_0}+\frac{(d^2\mathbf{Q}_\omega/dt^2)\cdot\mathbf{n}}{6c^2R_0}+\frac{(d\mathbf{m}_\omega/dt)\times\mathbf{n}}{cR_0}\right], \quad (3)$$

where $\mathbf{n}$ is the radial unit vector at the observation point, and $\mathbf{k}=\mathbf{n}\omega/c$. The quantities $\mathbf{d}_\omega$, $\mathbf{Q}_\omega$, $\mathbf{n}_\omega$ are respectively the ωth Fourier components of the electric dipole moment, the electric quadrupole moment, and the magnetic dipole moment of the radiating systems. These moments are given by

$$\mathbf{d}=e\left[\left(1-\frac{Z}{A}\right)\Big/\left(1+\frac{1}{A}\right)\right]\mathbf{r},$$

$$\mathbf{Q}=e\left[\left(1+\frac{Z}{A^2}\right)\Big/\left(1+\frac{1}{A}\right)^2\right](3\mathbf{rr}-r^2), \quad (4)$$

$$\mathbf{m}=e\left[\left(1+\frac{Z}{A^2}\right)\Big/\left(1+\frac{1}{A}\right)^2\right]\frac{1}{2c}(\mathbf{r}\times\dot{\mathbf{r}}),$$

where $\mathbf{r}$ is the vector pointing from the nucleus to the proton. The intensity of radiation emitted into the solid angle $d\Omega$, with frequency between ω and $\omega+\Delta\omega$ is, to the multipole orders considered, given by

$$\begin{aligned} d\mathcal{E}_\omega &= (\omega^2/c)\,|\mathbf{n}\times\mathbf{A}_\omega|^2R_0^2d\Omega\Delta\omega \\ &= (\omega^4/c^3)\{|\mathbf{d}_\omega\times\mathbf{n}|^2-(\omega/3c)\,\mathrm{Im}(\mathbf{d}_\omega\cdot\mathbf{Q}_\omega{}^*\cdot\mathbf{n}) \\ &\quad +2\,\mathrm{Re}[\mathbf{d}_\omega\cdot(\mathbf{m}_\omega{}^*\times\mathbf{n}]\}d\Omega\Delta\omega, \end{aligned} \quad (5)$$

where an asterisk indicates complex conjugate, and $|\mathbf{d}_\omega|^2=(\mathbf{d}_\omega{}^*\cdot\mathbf{d}_\omega)$. If we now consider the actual problem of the scattering of a uniform monoenergetic proton beam, we must average $d\mathcal{E}_\omega$ over the different orientations of the moments in the beam; i.e., we must average over the different azimuths of the planes of the proton orbit. We are mainly interested in the electric dipole term, because eventually we compare with experiments in which the radiation is emitted at 90° to the direction of the incident beam, and it can be easily shown that at 90°, the cross terms between it and the quadrupole and the magnetic dipole terms vanish, upon averaging over the incident proton beam. The higher moments

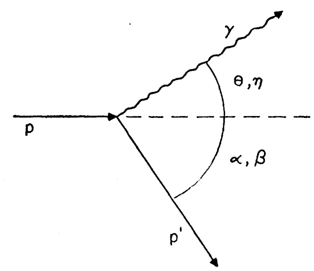

FIG. 1. Definition of angles. The proton is scattered through polar angle α and azimuthal angle β, emitting bremsstrahlung at polar angle θ and azimuthal angle η.

will contribute at angles other than 90°, but their contribution is expected to be small compared to the dipole term $((r_m/\lambda)<0.1)$. After averaging over the incident protons in the beam, we have for the intensity of dipole radiation:

$$d\mathcal{E}_\omega=(\omega^4/c^3)[\tfrac{2}{3}|\mathbf{d}_\omega|^2+\tfrac{1}{2}(3\cos^2\theta-1) \times(\tfrac{1}{3}|\mathbf{d}_\omega|^2-|d_{\omega x}|^2)]d\Omega\Delta\omega, \quad (6)$$

where θ denotes the angle between $\mathbf{n}$ and the incident direction, and $d_{\omega x}$ is the component of $\mathbf{d}_\omega$ along the incident direction. Integrating over proton impact parameters[3] and dividing by $\hbar\omega$, we obtain for the differential cross section for bremsstrahlung in frequency interval $(\omega,\omega+\Delta\omega)$ emitted into the solid angle $d\Omega$:

$$\sigma_\omega d\Omega=(\omega^4/c^3)(\Delta\omega/\hbar\omega)d\Omega\int_0^\infty 2\pi\rho d\rho \times[\tfrac{2}{3}|\mathbf{d}_\omega|^2+\tfrac{1}{2}(3\cos^2\theta-1)(\tfrac{1}{3}|\mathbf{d}_\omega|^2-|d_{\omega x}|^2)], \quad (7)$$

where ρ is the impact parameter of an incident proton.

We present several steps here in the evaluation of Eq. (7) in order to exhibit the impact parameters and eccentricities of the orbits which contribute most of the radiation. To express $\mathbf{d}_\omega$ as a function of the impact parameter, ρ, we parametrize the hyperbolic orbit of the proton as follows[3]:

$$\begin{aligned} x&=-a(\cosh\xi+\epsilon), \\ y&=a(\epsilon^2-1)^{\frac{1}{2}}\sinh\xi, \\ t&=(\xi+\epsilon\sinh\xi)/\omega_0, \end{aligned} \quad (8)$$

where

$$\begin{aligned} \omega_0&=(Ze^2/ma^3)^{\frac{1}{2}}, \quad m=\text{proton mass}, \\ a&=Ze^2/2E \quad (E=\text{incident proton energy}), \end{aligned} \quad (9)$$

and

$$\epsilon=[1+(\rho/a)^2]^{\frac{1}{2}} \quad (10)$$

is the eccentricity of the orbit. The geometric significance of these quantities are illustrated in Fig. 2. The angle ϕ between one asymptote and the x-axis is given by

$$\cos\phi=1/\epsilon. \quad (11)$$

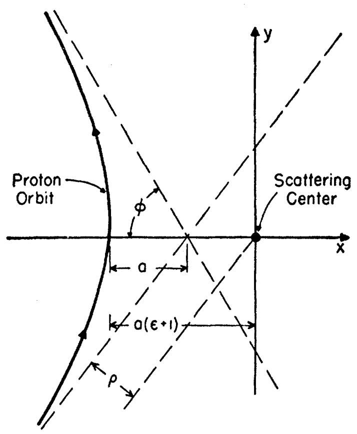

FIG. 2. Classical orbit for a proton in the Coulomb field of a fixed positively charged scattering center, showing impact parameter ρ, distance of closest approach $a(1+\epsilon)$, and asymptote angle ϕ.

From Eq. (4) we see that $\mathbf{d}_\omega = e'\mathbf{r}_\omega$, where we introduce the reduced charge:

$$e' = e\left(1-\frac{Z}{A}\right)\Big/\left(1+\frac{1}{A}\right), \tag{12}$$

and the Fourier transform of $\mathbf{r}$:

$$\mathbf{r}_\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathbf{r} e^{i\omega t} dt. \tag{13}$$

To conform to the coordinate system used in Eq. (7), the orbit as given by Eq. (8) must be rotated through an angle ϕ so that the proton is incident along the x-direction. We therefore get

$$\begin{aligned} d_{\omega x} &= d_{\omega x}' \sin\phi - d_{\omega y}' \cos\phi, \\ d_{\omega y} &= d_{\omega x}' \cos\phi + d_{\omega y}' \sin\phi, \end{aligned} \tag{14}$$

where $\sin\phi = (\epsilon^2-1)^{\frac{1}{2}}/\epsilon$, $\cos\phi = 1/\epsilon$, and

$$\begin{aligned} d_{\omega x}' &= (e'/2\pi)\int_{-\infty}^{+\infty} x e^{i\omega t} dt, \\ d_{\omega y}' &= (e'/2\pi)\int_{-\infty}^{+\infty} y e^{i\omega t} dt, \end{aligned} \tag{15}$$

and x, y, t are given by Eq. (8). These integrals can be evaluated to give

$$\begin{aligned} d_{\omega y}' &= \frac{e'a}{i\pi\omega}\frac{(\epsilon^2-1)^{\frac{1}{2}}}{\epsilon}\exp\left(-\frac{\pi}{2}\frac{\omega}{\omega_0}\right) K_{i\omega/\omega_0}(\epsilon\omega/\omega_0), \\ d_{\omega x}' &= -\frac{e'a}{i\pi\omega}\exp\left(-\frac{\pi}{2}\frac{\omega}{\omega_0}\right) K_{i\omega/\omega_0}'(\epsilon\omega/\omega_0), \end{aligned} \tag{16}$$

where $K_n(x)$ is related to the Hankel function of the first kind,[6] of imaginary argument, and $K_n'(x)$, its derivative with respect to x. We thus obtain, by substituting Eq. (16) into Eq. (14):

$$\begin{aligned} d_{\omega x} &= \frac{e'a}{\pi\omega} e^{-\frac{1}{2}\pi p}\frac{(\epsilon^2-1)^{\frac{1}{2}}}{\epsilon}\left[-K_{ip}'(p\epsilon) + \frac{i}{\epsilon}K_{ip}(p\epsilon)\right], \\ d_{\omega y} &= \frac{e'a}{\pi\omega} e^{-\frac{1}{2}\pi p}\frac{1}{\epsilon}\left[-K_{ip}'(p\epsilon) - \frac{i(\epsilon^2-1)}{\epsilon}K_{ip}(p\epsilon)\right], \\ |\mathbf{d}_\omega|^2 &= \frac{e'^2a^2}{\pi^2\omega^2} e^{-\pi p}\left\{[K_{ip}'(p\epsilon)]^2 + \frac{\epsilon^2-1}{\epsilon^2}[K_{ip}(p\epsilon)]^2\right\}, \end{aligned} \tag{17}$$

where $p=\omega/\omega_0$. To obtain the cross section, Eq. (17) must be substituted into Eq. (7). The integration in Eq. (7) over the impact parameter ρ can be converted into an integral over the eccentricity ϵ by the relation $\epsilon = [1+(\rho/a)^2]^{\frac{1}{2}}$. Thus we obtain

$$\sigma_\omega d\Omega = \left[A(\omega) + B(\omega)\frac{3\cos^2\theta - 1}{2}\right]\frac{\Delta\omega}{\hbar\omega} d\Omega, \tag{18}$$

where

$$A(\omega) = \frac{4}{3\pi}\frac{e'^2\omega^2a^4}{c^3} e^{-\pi p}\int_1^\infty \left\{[K_{ip}'(p\epsilon)]^2 + \frac{\epsilon^2-1}{\epsilon^2}[K_{ip}(p\epsilon)]^2\right\}\epsilon d\epsilon,$$

$$B(\omega) = \tfrac{1}{2}A(\omega) - \frac{2}{\pi}\frac{e'^2\omega^2a^4}{c^3} e^{-\pi p}\int_1^\infty \left\{[K_{ip}'(p\epsilon)]^2 + \left(\frac{\epsilon^2-1}{\epsilon}\right)^2[K_{ip}(p\epsilon)]^2\right\}\frac{d\epsilon}{\epsilon}.$$

At $\theta = 90°$, we obtain

$$\sigma_\omega(90°) = A_0 p^2 \int_1^\infty \frac{d\epsilon}{\epsilon}\left\{(\epsilon^2+1)K_{ip}'^2 + \frac{(\epsilon^2-1)(2\epsilon^2-1)}{\epsilon^2}K_{ip}^2\right\}, \tag{19}$$

where

$$A_0 = e^{-\pi p}\left(\frac{\Delta(\hbar\omega)}{\pi\hbar\omega}\right)\left[\left(1-\frac{Z}{A}\right)\Big/\left(1+\frac{1}{A}\right)\right]^2 \times\left(\frac{e^2}{\hbar c}\right)\left(\frac{2E}{mc^2}\right)\left(\frac{Ze^2}{2E}\right)^2. \tag{20}$$

Substituting the values

$$Z=50,\quad E=2\text{ Mev},\quad \hbar\omega=150\text{ kev},\quad \Delta\hbar\omega=3\text{ kev}, \tag{21}$$

we find

$$\begin{aligned} p=\omega/\omega_0 &= 0.206, \quad a=1.80\times10^{-12}\text{ cm}, \\ \omega_0 &= 1.10\times10^{21}\text{ sec}^{-1}, \quad A_0 = 1.09\times10^{-31}\text{ cm}^2. \end{aligned} \tag{22}$$

[6] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, 1948), Sec. 3.7.

Numerically integrating Eq. (19), we obtain the cross section result quoted in Eq. (2). For smaller values of $p\cong 0.1$ it is possible to simplify Eq. (19) to the approximate form

$$\sigma_\omega(90°)\cong A_0[\log(2/\gamma p)+1],$$

with $\gamma=1.781\cdots$, the Euler constant.

In concluding this discussion of the classical problem, we note that in the integration of Eq. (19), most of the contributions to the cross section results from orbits of eccentricities $1\leq\epsilon\lesssim 2$. Protons moving in such parabolic or nearly parabolic orbits approach to within $(3.6–6.2)\times 10^{-12}$ cm of the nucleus. This is well within the K-shell radius, (10^{-10} cm) of the tin atom. For this reason, we turn to quantum-mechanical rather than shielding corrections as the source of the discrepancy between Eqs. (2) and (1).[7]

III. QUANTUM-MECHANICAL CALCULATION

The quantum-mechanical calculation of the differential cross section for the dipole emission of a quantum of frequency ω into the solid angle $d\Omega$ has been carried out by Sommerfeld.[4] The matrix element which induces dipole transition between the continuum Coulomb states of the proton with emission of bremsstrahlung has the familiar form $\mathbf{M}=\int\psi_2{}^*\mathbf{r}\psi_1 d\tau$.

The cross section for the emission of a quantum of polarization $\mathbf{e}$ in the frequency interval $(\omega,\omega+\Delta\omega)$ into a solid angle element $d\Omega$, with the proton recoiling into solid angle element $d\Omega_k$, is directly:

$$\sigma(k_1,k_2,\omega,\mathbf{e}_k)=\left(\frac{e^2}{\hbar c}\right)\left[\left(1-\frac{Z}{A}\right)\Big/\left(1+\frac{1}{A}\right)\right]^2\frac{1}{16\pi^4}\left(\frac{k_2}{k_1}\right)\frac{m^2\omega^3}{\hbar^2c^2}\times|\mathbf{e}_k\cdot\mathbf{M}|^2\Delta\omega d\Omega_k d\Omega,\quad(23)$$

where k_1 and k_2 are the incident and final proton wave numbers, and the continuum wave functions in $\mathbf{M}$ are taken to be normalized in a unit volume. Integrating over the recoil proton solid angle and summing over photon polarizations, we obtain the differential cross section in dipole bremsstrahlung in the frequency interval $(\omega,\omega+\Delta\omega)$:

$$\sigma(\theta)d\Omega=\left(\frac{e^2}{\hbar c}\right)\left[\left(1-\frac{Z}{A}\right)\Big/\left(1+\frac{1}{A}\right)\right]^2\times\frac{1}{8\pi^3}\left(\frac{k_2}{k_1}\right)\frac{m^2\omega^4}{\hbar^2c^2}\left(\frac{\Delta\hbar\omega}{\hbar\omega}\right)d\Omega\int_0^\pi d\alpha\,\sin\alpha\times\{\tfrac{2}{3}|\mathbf{M}|^2+\tfrac{1}{2}(3\cos^2\theta-1)[\tfrac{1}{3}|\mathbf{M}|^2-|M_x|^2]\}.\quad(24)$$

[7] A classical calculation of the screening correction which treats the difference between a shielded and an unshielded Coulomb potential as a small perturbation has been carried through. For a shielding length as given by the Fermi-Thomas model, the correction is quite negligible (<0.1 percent).

We have taken the incident direction of the proton to be the x-axis. The angle θ is that between the photon and the x-axis, and α is the angle between the final proton and the x-axis, (Fig. 1). Angle α is related to the eccentricity of the corresponding classical orbit by

$$\sin\alpha=2(\epsilon^2-1)^{\frac{1}{2}}/\epsilon^2.\quad(25)$$

The dipole matrix elements have been evaluated by Sommerfeld:

$$M_x=C[(n_2-n_1\cos\alpha)F+(1-\cos\alpha)(1-x)F'](1-x)^{-n_1-n_2-1},$$

$$\left.\begin{matrix}M_y\\M_z\end{matrix}\right\}=C\left\{\begin{matrix}\cos\beta\\\sin\beta\end{matrix}\right\}\sin\alpha[n_1F+(1-x)F']\times(1-x)^{-n_1-n_2-1},\quad(26)$$

where F is the hypergeometric function and F' is the derivative of F with respect to its argument:

$$\begin{aligned}F&=F(-n_1,-n_2,1;x)\\F'&=dF/dx=n_1n_2F(1-n_1,1-n_2,2;x),\end{aligned}\quad(27)$$

with

$$\begin{aligned}n_1&=Ze^2m^2/ik_1\hbar^2,\\n_2&=Ze^2m^2/ik_2\hbar^2,\\x&=-(4k_1k_2\sin^2\alpha/2)/(k_1-k_2)^2,\end{aligned}\quad(28)$$

and

$$C=-16\pi e^{-i\pi n_1}\frac{k_1k_2}{(k_1+k_2)^2(k_1-k_2)^4}\left(\frac{k_1+k_2}{k_1-k_2}\right)^{n_1+n_2}.\quad(29)$$

The integration of Eq. (24) cannot be effected in closed form. Sommerfeld,[4] Scherzer,[8] Weinstock[9] (and others) have given approximate expressions, valid for conditions that do not prevail in the experiments under consideration here.

The problem here reduces to one of finding a suitable representation for the hypergeometric function, F, that permits a series expansion in powers of $\hbar$. We achieve this by transforming the independent variable from x to $1/x$ with the formula for the analytic continuation of the hypergeometric function:

$$F(a,b,c;x)=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-x)^{-a}\times F(a,1-c+a,1-b+a;x^{-1})+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-x)^{-b}\times F(b,1-c+b,1-a+b;x^{-1}).\quad(30)$$

Noting that both n_1 and $n_2\rightarrow\infty$ as $\hbar\rightarrow 0$,

$$F(a,b,c;x)\xrightarrow[|b|\rightarrow\infty]{|a|\rightarrow\infty}\Gamma(c)\left(\frac{x}{ab}\right)^{\frac{1}{2}(c-1)}\left\{I_{c-1}(2(ab/x)^{\frac{1}{2}})+\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{ab}{x}\right)I_{c+1}(2(ab/x)^{\frac{1}{2}})+O\left(\frac{1}{a^2},\frac{1}{b^2}\right)\right\},\quad(31)$$

[8] O. Scherzer, Ann. Physik 13, 137 (1932).
[9] R. Weinstock, Phys. Rev. 61, 584 (1946).

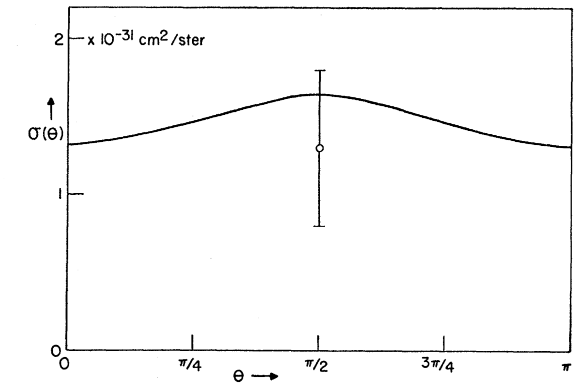

FIG. 3. Angular distribution of x-rays in a 3-kev interval about 150 kev for 2-Mev protons incident on a tin nucleus. The experimental point is that of Mark *et al.* (reference 2).

where

$$I_\nu(x)=\exp\left(\frac{\pi\nu}{2i}\right)J_\nu(e^{\frac{1}{2}i\pi}x)$$

$$=\left(\frac{x}{2}\right)^\nu\sum_{m=0}^{\infty}\frac{(x/2)^{2m}}{m!\Gamma(\nu+m+1)}. \quad (32)$$

Note that the new independent variable has a classical limit:

$$\left(\frac{ab}{x}\right)^{\frac{1}{2}}=\left(\frac{n_1n_2}{x}\right)^{\frac{1}{2}}\underset{\hbar\to 0}{\longrightarrow}\tfrac{1}{2}p\epsilon\left(1-\frac{\hbar\omega}{4E}\right)+O(\hbar^2).$$

Here ϵ is the eccentricity of the classical orbit as defined in Eq. (10), and $p=\omega/\omega_0$ is the same as that defined earlier in Eq. (22).

Carrying out this expansion in powers of $\hbar$ we obtain, for example,

$$F(1+in,\, in,\, 2-ip;\, x^{-1})\underset{\hbar\to 0}{\longrightarrow}\Gamma(2-ip)(\tfrac{1}{2}p\epsilon)^{-1+ip}$$

$$\times\left\{I_{1-ip}(p\epsilon)-\frac{i}{n}\frac{p\epsilon}{2}[\tfrac{1}{2}p\epsilon I_{1-ip}(p\epsilon)-I_{2-ip}(p\epsilon)]\right\} \quad (33)$$

$$F(i\lambda n,\, -1+i\lambda n,\, ip;\, x^{-1})\underset{\hbar\to 0}{\longrightarrow}\Gamma(ip)(\tfrac{1}{2}p\epsilon)^{1-ip}$$

$$\times\left\{I_{ip-1}(p\epsilon)-\frac{i}{n}\frac{p\epsilon}{2}[\tfrac{1}{2}p\epsilon I_{ip-1}(p\epsilon)-I_{ip}(p\epsilon)]\right\},$$

where $n=-in_1$, $\lambda n=-in_2$, $\lambda=(k_1/k_2)$.

The expansion parameter is seen to be

$$p/n=\hbar\omega/2E. \quad (34)$$

Making use of the identity[6]

$$[I_{-\nu}(x)-I_\nu(x)]=(2/\pi)(\sin\pi\nu)K\nu(x),$$

we now obtain

$$F(1+in,\, i\lambda n,\, 2;\, x)\underset{\hbar\to 0}{\longrightarrow}\frac{2x^{-1-in}(\frac{1}{2}p\epsilon)^{-1+ip}}{\Gamma(i\lambda n)\Gamma(1-in)}\left\{K_{ip-1}(p\epsilon)\right.$$

$$\left.-\frac{i}{n}\frac{p\epsilon}{2}[\tfrac{1}{2}p\epsilon K_{ip-1}(p\epsilon)+K_{ip-2}(p\epsilon)]\right\}. \quad (35)$$

Finally, we have

$$\left.\begin{matrix}|M_y|^2\\|M_z|^2\end{matrix}\right\}=\left\{\begin{matrix}\cos^2\beta\\\sin^2\beta\end{matrix}\right\}\frac{e^{-\pi p}}{p^2}\frac{(2\pi)^2(\epsilon^2-1)}{n^2}\left(\frac{n}{k_1}\right)^8\left(1-\frac{4p}{n}\right)\epsilon^2$$

$$\times\left[\left(K_{ip}'^2+\frac{1}{\epsilon^2}K_{ip}^2\right)-\frac{2p}{n}\left(K_{ip}'^2+\frac{2-\epsilon^2}{2\epsilon^2}K_{ip}^2\right)\right], \quad (36)$$

$$|M_x|^2=\frac{e^{-\pi p}}{n^2p^2}\left(\frac{n}{k_1}\right)^8\left(1-\frac{4p}{n}\right)(2\pi)^2\epsilon^2$$

$$\times\left\{K_{ip}'^2+\left(\frac{\epsilon^2-1}{\epsilon}\right)^2K_{ip}^2-\frac{p\epsilon}{n}\left(\frac{2-\epsilon^2}{\epsilon}K_{ip}'^2\right.\right.$$

$$\left.\left.+\frac{2(\epsilon^2-1)^2}{\epsilon^3}K_{ip}^2\right)\right\}, \quad (37)$$

$$|\mathbf{M}|^2=\frac{(2\pi)^2e^{-\pi p}}{n^2p^2}\left(\frac{n}{k_1}\right)^8\left(1-\frac{5p}{n}\right)$$

$$\times\epsilon^4\left[K_{ip}'^2+\frac{\epsilon^2-1}{\epsilon^2}K_{ip}^2\right]. \quad (38)$$

The differential cross section is thereby

$$\sigma(\theta)=A_0\left(1-\frac{5\hbar\omega}{2E}\right)\left\{\frac{4}{3}\left(1-\frac{\hbar\omega}{2E}\right)J_1\right.$$

$$\left.+(3\cos^2\theta-1)\left(J_2+\frac{\hbar\omega}{2E}J_3\right)\right\}, \quad (39)$$

where

$$J_1=\int_p^\infty\left[K_{ip}'^2(x)+\frac{x^2-p^2}{x^2}K_{ip}^2(x)\right]xdx,$$

$$J_2=\int_p^\infty\left[\left(\frac{x^2}{3}-p^2\right)K_{ip}'^2(x)\right.$$

$$\left.+\frac{(x^2-p^2)(\frac{2}{3}x^2-p^2)}{x^2}K_{ip}^2(x)\right]\frac{dx}{x}, \quad (40)$$

$$J_2=\int_p^\infty\left[\left(2p^2-\frac{4}{3}x^2\right)K_{ip}'^2(x)\right.$$

$$\left.+\frac{(x^2-p^2)(5x^2-6p^2)}{3x^2}K_{ip}^2(x)\right]\frac{dx}{x}.$$

We see that Eqs. (39) and (40) reduce to the classical formulas of the preceding section in the limit $(\hbar\omega/E)\to 0$. Numerical integration of these formulas for the parameter values given in Eqs. (21) and (22) gives a differential cross section at 90° of

$$\sigma(90°)=1.65\times 10^{-31}\text{ cm}^2/\text{sterad},$$

in satisfactory agreement with the experimental result in Eq. (1). We have also computed an angular distribution curve for the emission of x-rays in a 3-kev energy interval about 150 kev for 2 Mev protons incident on a tin target. The result given in Fig. 3 is nearly isotropic in agreement with preliminary[2] results. For angles other than 90° there will be corrections to this curve due to the dipole-quadrupole cross terms in Eq. (5), but these are expected to alter the results by less than 10 percent. The total cross section is, from Eqs. (39) and (40),

$$\sigma_{\text{tot}}(k_1,k_2,\omega)=(16\pi/3)A_0(1-3\hbar\omega/E)J_1 \\ =18\cdot 9\times 10^{-31}\ \text{cm}^2.$$

IV. CONCLUSION

The quantum correction to the classical orbit treatment of dipole radiation emitted by a beam of protons scattered in a coulomb field has been calculated and found to agree with experiments of Mark *et al.*[2] This correction has been expressed in terms of the inelasticity of the scattered proton, ($\hbar\omega/E$). It serves to decrease the classical prediction by ~25 percent for the experiments under discussion here. This reduction seems at first to be surprisingly large in view of the fact that the proton radiates only $\hbar\omega/E=7.5$ percent of its energy. However, we can make it reasonable by recalling that the important orbits in the classical picture are parabolic or nearly parabolic. The damping effect of the bremsstrahlung on the motion of the proton, which is neglected in the classical treatment, causes it to be slowed down and repelled further away from the Coulomb field source, thereby reducing the effective dipole moment by an appreciable amount.

Other corrections due to retardation effects, electronic shielding, and finite nuclear size can be neglected for the energies studied in this work.

We wish to thank Dr. Hans Mark for informing us of his experimental results prior to publication and thereby arousing our interest in this problem.

Note added in proof.—We have learned (private communication) that Professor L. C. Biedenharn of The Rice Institute has made the reduction to the classical orbit calculation expressed in Eq. (35) in connection with his studies on the electric excitation problem.

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Angular Correlation of the Gamma Rays of Ba^{134}

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The angular correlations have been measured between the following pairs of gamma rays from Ba^{134}: 605 kev–797 kev, 1368 kev–605 kev, and 570 kev–605 kev. These measurements indicate that the spins of the states associated with these gamma rays are 4, 4, 2, 0. The 605, 797, and 1368-kev gamma rays are pure quadrupole while the 570-kev gamma ray is a mixture of 94 percent quadrupole and 6 percent dipole.

I. INTRODUCTION

The angular correlation of the gamma rays following the beta decay of the 2.3-year isomer of Cs^{134} has been investigated by a number of authors.[1–4] While the complete decay scheme is very complex,[5,6] the main features are shown in Fig. 1. The gamma-ray spectrum as observed is shown in Fig. 2. It should be noted that the remaining lines are too weak to be observed. The first investigators noted the similarity between the shape of the curve obtained from their measurements of the overall angular correlation and the shape of the angular correlation function obtained from the gamma rays following the beta decay of Co^{60}. The cases differed in that the asymmetry for the Ba^{134} was approximately 12 percent while that of Co^{60} was 17 percent. Since the majority of the coincidence counts were due to the 797-kev and 605-kev gamma rays, it was proposed that these two gamma rays gave rise to a basic quadrupole-quadrupole correlation. Since Ba^{134} is an even-even nucleus (56 protons-78 neutrons) the ground state is assumed to have a spin of 0 with even parity. Thus, as is in the case of Co^{60}, spins of 2 and 4 were assigned to the first and second excited states respectively with both radiations quadrupole. Robinson and Madansky[3] measured the correlation of the lower two gamma rays by demanding that they also be in coincidence with the high-energy beta ray. From this measurement and from a measurement of the over-all correlation, spins of 5, 4, 2, 0 were assigned to the important states with the radiation being dipole, quadrupole, quadrupole respectively. However because of the discrepancies between the over-all correlation and the above assignment, it was decided to reinvestigate the angular correlations in greater detail.

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