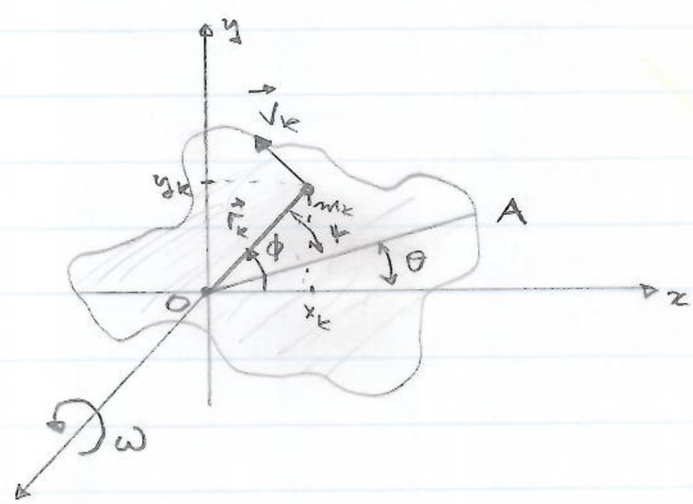
Rigid Body Motion I

A **RIGID BODY** is defined as a system consisting of a large number of point masses, called particles, such that the distances between pairs of point masses remain constant even when the body is in motion or under the action of external forces

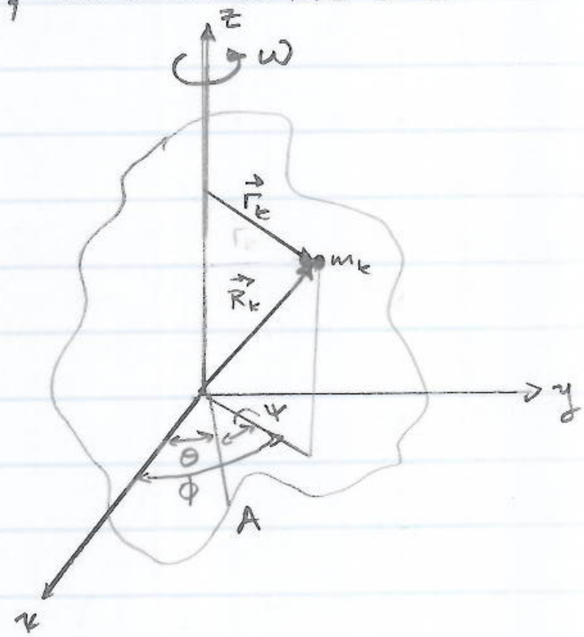
Forces that maintain constant distances between different pairs of point masses are internal forces and are called FORCES OF CONSTRAINT. Such forces come in pairs and Newton's Third Law in the strong form, that is, they are equal and opposite and act along the same line of action

Rotation about an Axis



Let us consider a rigid body that rotates about a fixed z-axis. The position of the body may be specified by an angle θ, which is between the line OA drawn on the body and the x-axis. Let us consider a mass, m_k

to be the representative particle located a distance $\vec{R}_k$ from the origin, moving with velocity $\vec{V}_k$ and angular velocity ω. The path of such a particle is a circle of radius $r_k = (x_k^2 + y_k^2)^{1/2}$ with its center on the z-axis. Let ψ be the angle between the direction of the line OA in the body and the radius r_k from the z-axis to the mass m_k. Since for a rigid body ψ is constant, as shown in the figure, $\dot{\phi} = \dot{\theta} - \dot{\psi}$



$$\dot{\phi} = \dot{\theta} = \omega$$

$$V_k = r_k \omega$$

or in vector notation

$$\vec{V}_k = \vec{\omega} \times \vec{r}_k$$

Then

$$\dot{x}_k = -V_k \sin\phi = -\omega y_k$$

$$\dot{y}_k = V_k \cos\phi = \omega x_k$$

$$\dot{z}_k = 0$$

$$V_k = r_k \omega = (\dot{x}_k^2 + \dot{y}_k^2)^{1/2}$$

For further calculations, we can use either rectangular (x, y, z) or cylindrical coordinates

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The kinetic energy of the rotating body about the z-axis is:

$$K = \sum_k \frac{1}{2} m_k v_k^2 = \frac{1}{2}\underbrace{\left[\sum_k m_k r_k^2\right]}_{I_z = \sum_k m_k (x_k^2 + y_k^2)} \omega^2$$

$$K = \frac{1}{2} I_z \omega^2 = \frac{1}{2} I_z \dot{\theta}^2$$

The quantity I_z is constant for a given RIGID BODY rotating about a given axis (z-axis in this case).

I_z : Moment of Inertia about the z-axis.

If the body is continuous

$$I_z = \iiint_{(body)} r^2 dm = \iiint_{(body)} r^2 \rho \, dV$$

Let us now calculate the angular momentum of the body about the z-axis

$$L = \sum_k r_k (m_k v_k) = \left[\sum_k m_k r_k^2\right] \omega$$

$$L = I_z \omega = I_z \dot{\theta}$$

The rate of change of angular momentum for any system is equal to the ~~total~~ EXTERNAL TORQUE

$$\Gamma_z = \frac{dL}{dt} = I_z \frac{d\omega}{dt} = I\ddot{\theta}$$

Rectilinear motion		Rotation about a fixed axis	
position:	x	angular position	θ
velocity:	$v = \dot{x}$	angular velocity	$\omega = \dot{\theta}$
acceleration:	$a = \ddot{x}$	angular acceleration	$\alpha = \ddot{\theta}$
force:	F	torque	Γ_z
mass:	m	moment of inertia	I_z
kinetic energy:	$K = \frac{1}{2}m\dot{x}^2$	kinetic energy	$K = \frac{1}{2}I_z\dot{\theta}^2$

potential energy:

$$U(x) = -\int_{x_s}^{x} F(x)\,dx$$

$$F(x) = -\frac{dU}{dx}$$

potential energy

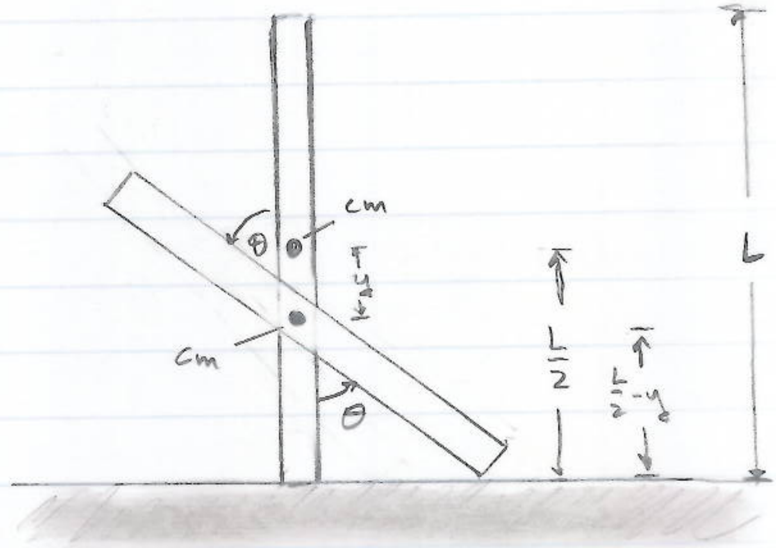
$$U(\theta) = -\int_{\theta_s}^{\theta} \Gamma_z(\theta)\,d\theta$$

$$\Gamma_z(\theta) = -\frac{dU}{d\theta}$$

linear momentum: $p = m\dot{x}$

angular momentum $L = I_z\dot{\theta}$

<u>Example</u> A stick of mass M and length L is initially at rest in a vertical position on a frictionless table as is shown in the figure. If the stick starts falling, find the speed of the center of mass as a function of the angle that the stick makes with respect to the vertical



$$K_i = 0 \qquad U_i = Mg\frac{L}{2}$$

$$K_f = K_{rot} + K_{trans} = \frac{1}{2}MV_y^2 + \frac{1}{2}I_o\omega_\theta^2$$
$$U_f = Mg\left(\frac{L}{2} - y\right)$$

Now $V_y = |\vec{\omega}\times\vec{r}\,| = \omega_\theta\left(\frac{L}{2}\right)\sin\theta \Rightarrow \omega_\theta = \dfrac{2V_y}{L\sin\theta}$

From the conservation of mechanical energy: $E_i = E_f$

$$Mg\frac{L}{2} = \frac{1}{2}MV_y^2 + \frac{1}{2}I_o\omega_\theta^2 + Mg\left(\frac{L}{2}-y\right)$$
$$= \frac{1}{2}MV_y^2 + \frac{1}{2}I_o\left(\frac{2V_y}{L\sin\theta}\right)^2 + Mg\left(\frac{L}{2}-y\right)$$
$$= \left(\frac{M}{2} + \frac{2I_o}{L^2\sin^2\theta}\right)V_y^2 + \frac{MgL}{2} - Mgy$$

or

$$\left(\frac{M}{2} + \frac{2I_o}{L^2\sin^2\theta}\right)V_y^2 - Mgy = 0 \qquad (1)$$

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We now must calculate the moment of inertia of a rod of mass M rotating about its center of mass

Let $dm = \lambda dx$ and $\lambda = \frac{M}{L}$

$$I_o = \int_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} x^2 dm = \int_{-L/2}^{+L/2} x^2 \lambda dx = \lambda \frac{x^3}{3}\Big|_{-L/2}^{+L/2} = \left(\frac{M}{L}\right)\left(\frac{L^3}{12}\right)$$

$$I_o = \frac{1}{12} M L^2 \qquad (2)$$

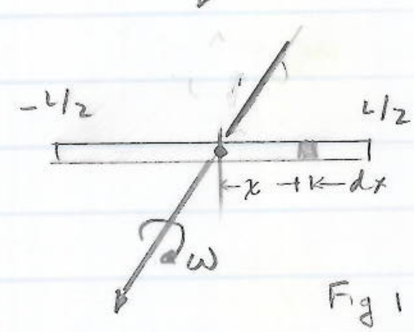

Fig 1

inserting (2) into (1)

$$\left(\frac{M}{2} + \frac{M}{6\sin^2\theta}\right) V_y^2 - Mgy = 0$$

$$\left(\frac{3\sin^2\theta - 1}{6\sin^2\theta}\right) V_y^2 = gy$$

$$\boxed{V_y = \left[\frac{(6\sin^2\theta)\, gy}{3\sin^2\theta - 1}\right]^{1/2}}$$

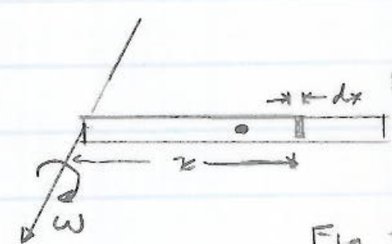

Fig 2

Now suppose we wish to find the moment of inertia about an axis perpendicular to the rod at one end (Fig 2)

$$I = \int_0^L x^2 dm = \int_0^L x^2 \lambda dx = \frac{1}{3} L^3 \lambda = \frac{1}{3} L^3 \left(\frac{M}{L}\right)$$

$$I = \frac{1}{3} M L^2.$$

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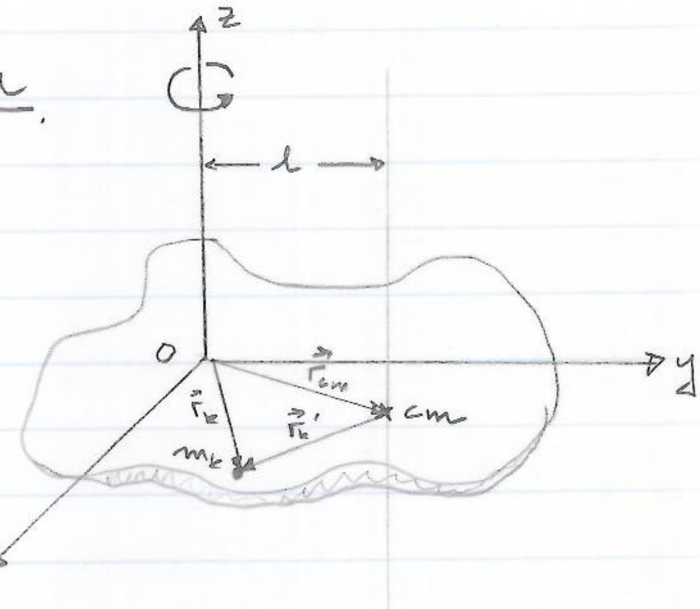
Parallel Axis Theorem



Consider a body rotating about an axis passing through O. By definition, the moment of inertia about an axis passing through O is

$$I_o = \sum_k m_k(x_k^2 + y_k^2) = \iiint_{(body)} (x^2 + y^2)\rho\, dV \qquad (1)$$

From the figure $\vec{r}_k = \vec{r}_{cm} + \vec{r}_k\,'$, where $\vec{r}_{cm}$ is the distance of the center of mass from the origin O, and $\vec{r}_k\,'$ is the relative coordinate of m_k wrt the cm.

(Dropping the subscript k):

$$(2) \qquad x^2 + y^2 = (x_{cm} + x')^2 + (y_{cm} + y')^2 = x_{cm}^2 + y_{cm}^2 + x'^2 + y'^2 + 2x_{cm}x' + 2y_{cm}y'$$

substituting (2) into (1) yields

$$(3) \quad I_o = \iiint_{(body)} (x'^2 + y'^2)\rho\, dV + (x_{cm}^2 + y_{cm}^2) \iiint_{(body)} \rho\, dV$$

$$+ 2x_{cm} \underbrace{\iiint_{(body)} x'\rho\, dV}_{x_{cm} \text{ wrt to cm.}} + 2y_{cm} \underbrace{\iiint_{(body)} y'\rho\, dV}_{y_{cm} \text{ wrt origin} = 0}$$

The first term on the right hand side of equation (3) is the moment of inertia about an axis parallel to the z-axis passing through the center of mass:

$$I_{cm} = \iiint (x'^2 + y'^2)\rho\, dV$$

The second term:

$$(x_{cm}^2 + y_{cm}^2)\iiint \rho\, dV = (x_{cm}^2 + y_{cm}^2) M = M\ell^2$$

The last two terms are zero by definition of the center of mass; that is, they simply locate the center of mass relative to itself. [They are proportional to x_{cm} and y_{cm} which we have taken to be the origin of the center of mass.

Thus combining the above we obtain:

$$\boxed{I_o = I_{cm} + M\ell^2}$$

<u>Parallel Axis Thm</u> The moment of inertia of a body about any given axis is the moment of inertia about a parallel axis through the center of mass, plus the moment of inertia about the given axis if all the mass of the body were located at the center of mass.

We see therefore

$$I_z = I_x + I_y$$

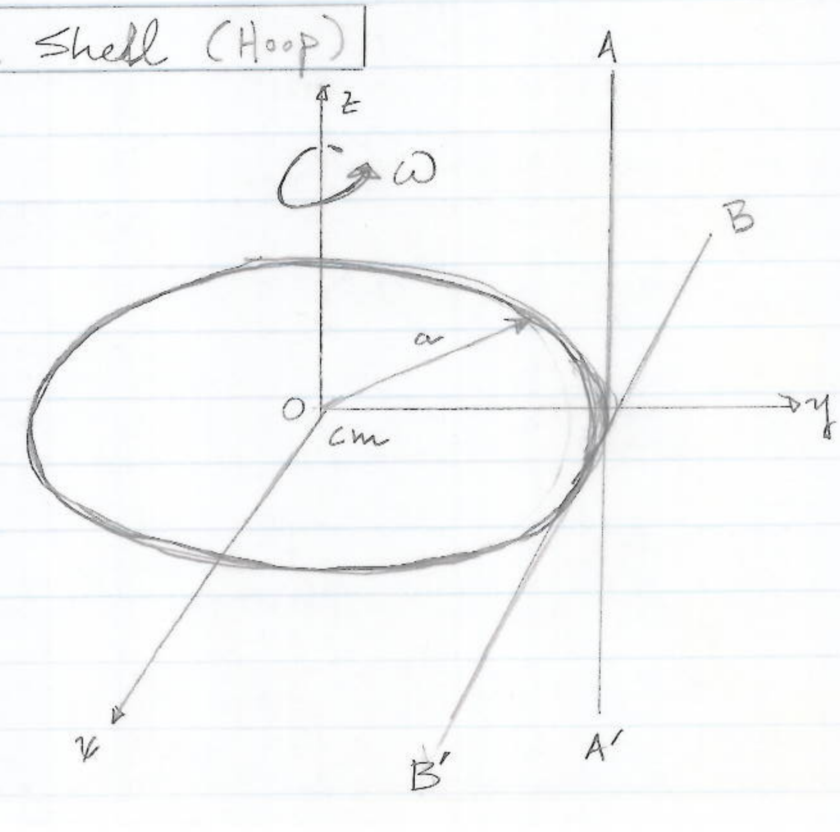
which is the

<u>Perpendicular Axis THEOREM</u>. The sum of the moments of inertia of a plane lamina about any two perpendicular axes in the plane of the lamina is equal to the moment of inertia about an axis that passes through the point of intersection and perpendicular to the plane of the lamina

Let's apply these theorems ....

$\boxed{\text{Cylindrical Shell (Hoop)}}$

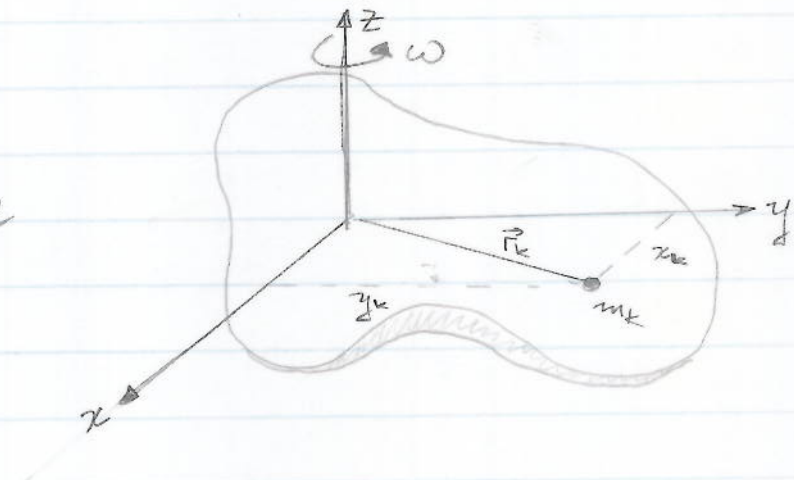
Moment of inertia for a hoop about three different axes: z-axis, AA′-axis, and BB′-axis.



Perpendicular Axis Theorem

A body whose mass is concentrated in a single plane is called a plane lamina. The perpendicular axis theorem is applicable to a plane lamina of any shape

Plane Lamina
in the x-y plane



Let us consider a rigid body in the form of a lamina in the x-y plane. For rotation about the z-axis, the moment of inertia about this axis is:

$$I_z = \sum m_k(x_k^2 + y_k^2) = \iiint (x^2+y^2)\rho \, dV$$

If the body were rotating about the x-axis, its moment of inertia about the x-axis (IT IS A THIN LAMINA $z=0$ HENCE NO z^2 TERM)

$$I_x = \iiint y^2 \rho \, dV$$

Similarly, the moment of inertia about the y-axis is

$$I_y = \iiint x^2 \rho \, dV$$

Consider a hoop or ring of mass M and radius a. All the mass is concentrated at a distance a from the axis. Therefore, the moment of inertia about the z-axis is

$$I_z = Ma^2$$

Now suppose that we wish to calculate the moment of inertia about the AA′ axis – that is perpendicular to the plane of the ring and parallel to the z-axis and passing through the edge of the ring as depicted above. The situation is no longer symmetrical. The direct calculation of the moment of inertia is NOT TRIVIAL. Making use of the parallel-axis theorem: $I_0 = I_{cm} + Ml^2$

$$I_{AA'} = I_z + Ma^2 = Ma^2 + Ma^2$$
$$= 2Ma^2$$

Next we proceed to calculate the moment of inertia of the ring about an axis in the plane of the ring, such as the x- or y-axis. From the symmetry of the ring, we have:

$$I_x = I_y$$

and applying the perpendicular-axis theorem

$$I_z = I_x + I_y = 2I_x = 2I_y.$$

This implies:

$$I_x = I_y = \frac{1}{2} M a^2$$

We can now apply the parallel-axis theorem to find the moment of inertia about the BB′ axis, which is in the plane of the ring and is tangent to the edge. Hence

$$I_{BB'} = I_x + Ma^2 = \frac{1}{2} M a^2 + Ma^2$$
$$= \frac{3}{2} M a^2$$

RADIUS of GYRATION

Often it is convenient to express the moment of inertia of a rigid body in terms of a distance k, called the **radius of gyration**:

$$I = Mk^2 \; ; \; k = \left[\frac{I}{M}\right]^{1/2}$$

That is, the radius of gyration is that distance from the axis of rotation where we may assume that ALL OF THE MASS of that body to be concentrated.

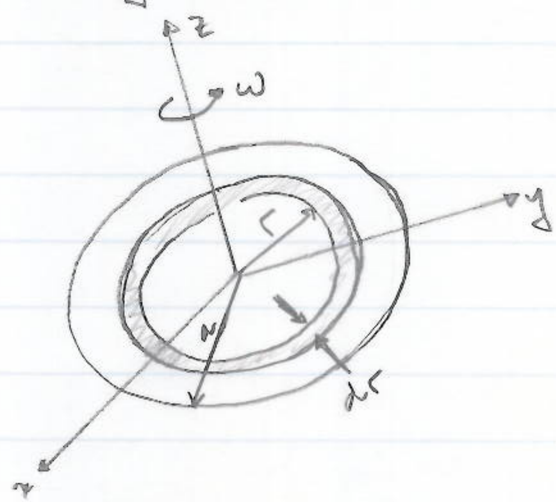
For example, the radius of gyration k for a thin rod about the axis of rotation passing through the center is

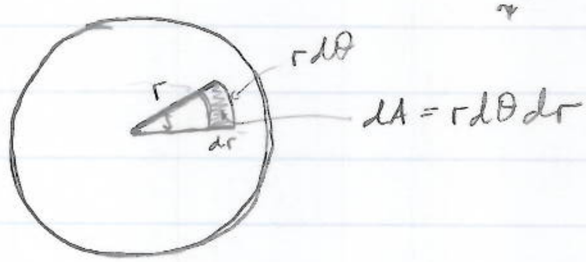
$$k = \left[\frac{I}{M}\right]^{1/2} = \left[\frac{\frac{1}{12} M a^2}{M}\right]^{1/2} = \frac{a}{\sqrt{12}} .$$

Let us find the moment of inertia for a few bodies...

Circular Disk - Solid Cylinder

Moment of inertia for a disk about an axis perpendicular to the plane of the disk.





$$I_z = \iint r^2 dm \quad ; \quad dm = \sigma dA$$
$$= \sigma (r dr d\theta)$$

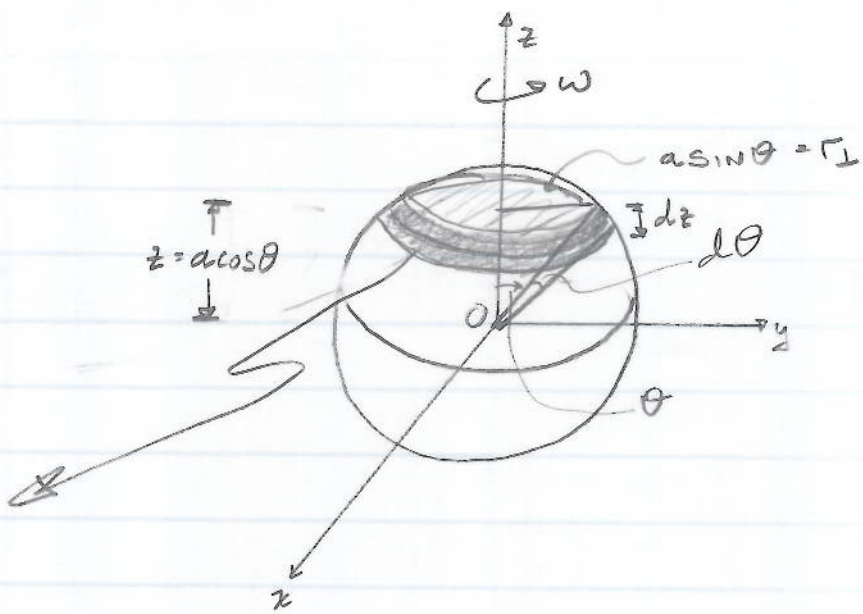
$$= \iint r^2 \sigma r dr d\theta = \sigma \int_0^a r^3 dr \int_0^{2\pi} d\theta$$

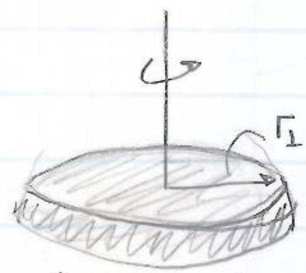
$$= \frac{M}{\pi a^2}(2\pi) \int_0^a r^3 dr =$$

$$\boxed{I_z = \frac{1}{2} M a^2}$$

R-14

Solid Sphere





"slice" of mass dm

The volume of the slice is $dV = \pi r^2 dz$

$$dV = \pi(a\sin\theta)^2\, d(a\cos\theta)$$

The sphere is made up of a series of slices. The slices are thin disks. The moment of inertia of such a disk about the z-axis is:

$$dI = \frac{1}{2}r^2 dm = \frac{1}{2}r^2(\rho\, dV)$$

$$= \frac{1}{2}(a\sin\theta)^2 \frac{M}{\frac{4}{3}\pi a^3}\pi(a\sin\theta)^2 d(a\cos\theta)$$

$$dI = -\frac{3}{8}Ma^2\sin^5\theta\, d\theta$$ or moment of inertia of the slice about the z-axis.

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The moment of inertia for the solid sphere is

$$I = \int_{\pi}^{0} dI = \frac{3}{8} M a^2 \int_{0}^{\pi} \sin^5\theta \, d\theta$$

$$\int \sin^n ax \, dx = -\frac{\sin^{n-1} ax \cos ax}{na} + \frac{n-1}{n} \int \sin^{n-2} ax \, dx \qquad (n>0)$$

$$\Rightarrow \boxed{I_z = \frac{2}{5} M a^2}$$

<u>Spherical</u> <u>Shell</u>

$dI = r^2 dm$ ← formed of a series of hoops of radius $a\sin\theta$ and width $a d\theta$

$M = \sigma(4\pi a^2)$; $dm = \sigma dA = \sigma 2\pi (a\sin\theta) a d\theta$

$$dI = (a\sin\theta)^2 \sigma (2\pi) a\sin\theta (a\, d\theta)$$

$$= \frac{M}{4\pi a^2} 2\pi a^4 \sin^3\theta \, d\theta = \frac{1}{2} M a^2 \sin^3\theta \, d\theta$$

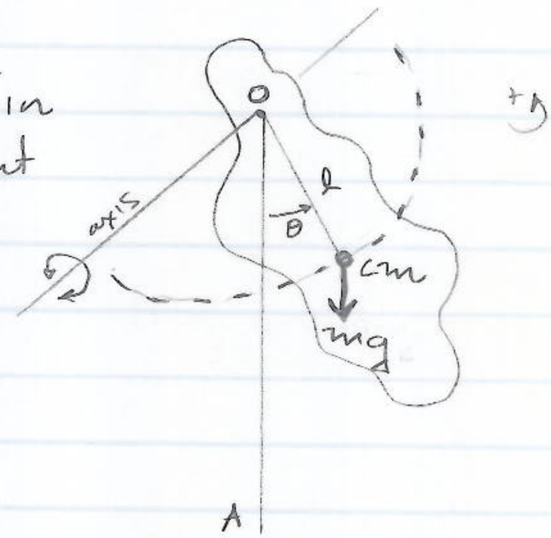
$$I = \int_{\pi}^{0} dI = \frac{1}{2} M a^2 \underbrace{\int_{\pi}^{0} \sin^3\theta \, d\theta}_{\underbrace{\int_{\pi}^{\pi/2} \sin^3\theta \, d\theta + \int_{\pi/2}^{0} \sin^3\theta \, d\theta}_{4/3}}$$

$$\Rightarrow \boxed{I_z = \frac{2}{3} M a^2}$$

PHYSICAL PENDULUM

A rigid body suspended and free to swing under its own weight about a fixed axis of rotation is known as a PHYSICAL PENDULUM or COMPOUND PENDULUM. The rigid body can be of any shape as long as the horizontal axis does not pass through the center of mass.

The pendulum swings in an arc of a circle about an axis of rotation that passes through O. Here O is the point of suspension,



The torque τ_o about the axis of rotation through O produced by the force Mg acting at point cm is

$$\tau_o = -Mg\,l\sin\theta$$

$$\tau_o = I\ddot{\theta} \qquad \text{(equation of motion, cf. NII)}$$

$$\Rightarrow \quad \ddot{\theta} + \frac{Mgl}{I}\sin\theta = 0$$

For the case of small oscillations, $\sin\theta \simeq \theta$ and hence

$$\ddot{\theta} + \frac{\text{M}gl}{I}\theta \simeq 0$$

This is the equation for simple Harmonic motion

$$\theta = \theta_0 \cos(\omega t + \varphi)$$

where the amplitude θ_0 and the phase angle φ are to be determined by the initial conditions. The angular frequency is

$$\omega^2 = \frac{\text{M}gl}{I}$$

and the period is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{\text{M}gl}}$$

If k is the radius of gyration for the moment of inertia about the axis of rotation through O, then

$$I = \text{M}k^2$$

or

$$T = 2\pi\sqrt{\frac{k^2}{gl}}$$

A simple pendulum of length k^2/l will have the same time period as that of a physical pendulum.

Let us take the moment of inertia of the rigid body about an axis passing through the center of mass cm and PARALLEL to the axis through O is I_{cm}. We can define the corresponding radius of gyration k_{cm} by.

$$I_{cm} = M k_{cm}^2$$

Making use of the parallel-axis theorem, we have:

$$I = I_{cm} + Ml^2$$

$$Mk^2 = Mk_{cm}^2 + Ml^2$$

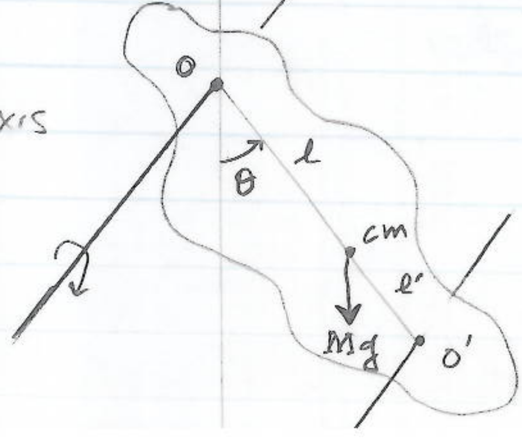
$$\Rightarrow \quad k^2 = k_{cm}^2 + l^2$$

The time period then is

$$T = 2\pi \sqrt{\frac{k_{cm}^2 + l^2}{gl}}$$

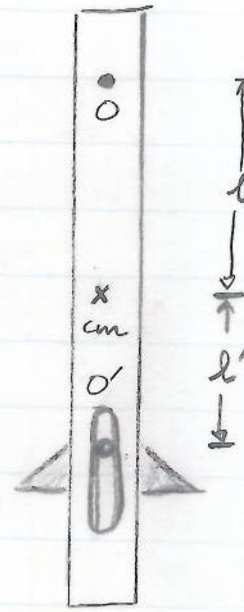
Let us now change the axis of rotation of this physical pendulum to a different position O′ at a distance l' from the center of mass cm.

The time period T' of oscillation about this new axis of rotation is



$$T' = 2\pi \sqrt{\frac{k_{cm}^2 + l'^2}{gl'}}$$

Let us now adjust l' so that



Kater's Pendulum

$$T = T'$$

$$\frac{k_{cm}^2 + l^2}{l} = \frac{k_{cm}^2 + l'^2}{l'}$$

which simplifies to

$$k_{cm}^2 = l l'$$

This is called the <u>center of oscillation</u> for point O in relation to point O'

$$T = 2\pi\sqrt{\frac{l+l'}{g}}$$

or

$$g = 4\pi^2\left(\frac{l+l'}{T^2}\right)$$

Such a physical pendulum is called Kater's reversible pendulum. This provides a very accurate way (1 part in 10^5) to measure the acceleration due to gravity.