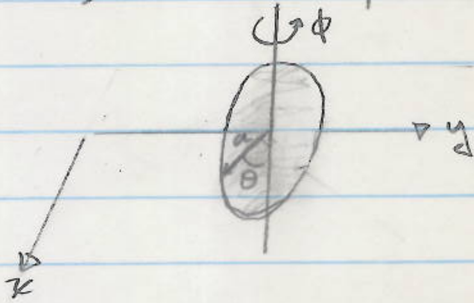
7-19

Lagrange's Equations with Undetermined Multipliers.

Let us consider a circular disk of radius a rolling over a perfectly rough (NO SLIPPING) horizontal plane

We require that the plane of the disk is perpendicular to the x-y plane.



To describe the configuration of the disk at any given instant we need four coordinates: x, y, θ, and ϕ. The coordinates x, y of the center of the disk (with $z = a$) describe the translational motion of the disk, and locate the point of contact of the disk with the plane. Angle θ describes the rotational motion of the disk about the center of mass. Angle ϕ describes the orientation of the disk wrt the x-axis; that is, it give the INSTANTANEOUS direction of the motion.

These coordinates are not all independent. Because of the constraints, the velocity $\vec{v}$ of the center of mass and $\dot{\theta}$ are related by

$$v = a\dot{\theta}$$

This results in the velocity components

$$\dot{x} = v\cos\phi = a\dot{\theta}\cos\phi$$
$$\dot{y} = -v\sin\phi = -a\dot{\theta}\sin\phi$$

These yield the following two equations of constraint

$$dx = a\,d\theta\cos\phi$$
$$dy = -a\,d\theta\sin\phi$$

Neither of these differential equations can be integrated to obtain two relations between x, y, and ϕ. Such constraints in which the differential equations are not integrable and are called:

NONHOLONOMIC CONSTRAINTS

This means that although the constraints relate the infinitesimal displacements, they do not dictate the relations among the coordinates themselves, i.e. the value of x and y (position) in NO WAY determine θ or ϕ (pitch and yaw), and vice versa.

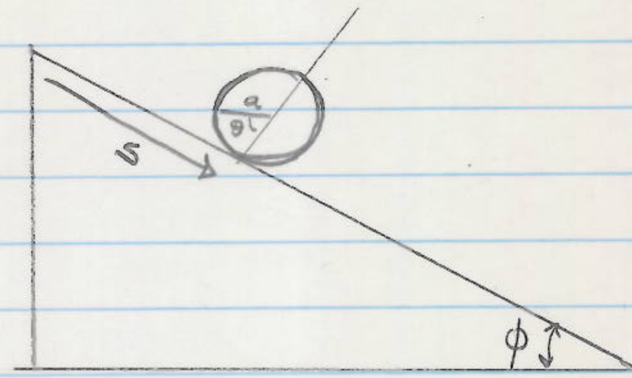
Suppose, however, the disk were constrained to roll along a prescribed curve. Let s measure the length of the path along this curve. Then $v = a\dot{\theta}$ can be integrated to yield

$$ds = a\,d\theta \Rightarrow s - a\theta = \text{constant}$$

7-21

We then have a condition representing a holonomic constraint ⇒ Hence a holonomic system

Let us consider the case of a disk rolling down an inclined plane, without slipping. The position can be located by two coordinates s and θ.



The velocities $\dot{s}$ and $\dot{\theta}$ are related by

$$\dot{s} = a\dot{\theta}$$
$$ds = a\,d\theta$$

$\Rightarrow \quad s - a\theta = \text{constant}$

Thus the system is holonomic with one equation of constraint, and only one coordinate is needed to describe the system.

⇒ If in a certain system, the constraint on the velocities can be integrated to give a relationship between the coordinates, the constraint, then, is HOLONOMIC

7-22

Let us pursue this matter further. Consider a constraint in the form

$$\sum_i A_i \dot{x}_i + B = 0 \qquad i = 1, 2, 3$$

In general, this is nonintegrable; hence it represents a nonholonomic constraint. But if A_i and B have the following form

$$A_i = \frac{\partial f}{\partial x_i}, \quad B = \frac{\partial f}{\partial t}, \quad f = f(x_i, t)$$

we then may write

$$\sum_i \frac{\partial f}{\partial x_i} \frac{dx_i}{dt} + \frac{\partial f}{\partial t} = 0$$

But this is just

$$\frac{df}{dt} = 0$$

and we may integrate to give

$$f(x_i, t) = \text{constant.}$$

So the constraint, then, is holonomic

In general, a constraint expressed as

$$\sum_{i=1} \frac{\partial f_l}{\partial q_i} dq_i + \frac{\partial f_l}{\partial t} dt = 0$$

is entirely equivalent to

$$f_l = f_l(q_i, t) = 0$$

There are certain advantages to expressing the constraints in differential form. We can incorporate the constraint relations (without first integrating them) into Lagrange's equations by means of the Lagrange undetermined multipliers

Suppose the constraints are expressed as

$$\sum_{l} \frac{\partial f_l}{\partial q_i} dq_i = 0 \qquad \begin{cases} i = 1, 2, \ldots, s \\ l = 1, 2, \ldots, m \end{cases}$$

Then Lagrange's equations take the form

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \sum_{l} \lambda_l(t) \frac{\partial f_l}{\partial q_k}$$

Here $\lambda_l(t)$ are the UNDETERMINED MULTIPLIERS and they represent the FORCES OF CONSTRAINT

If this is true then it must follow that

$$\lambda_l \sum_i \frac{\partial f}{\partial q_i} dq_i = 0$$

where λ_l are the undetermined coefficients called the LAGRANGE MULTIPLIERS

It is seen from Hamilton's Principle that

$$\int \sum_i \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} \right) dq_i = 0$$

And by the same process

$$\int \sum_{l,i} \lambda_l \frac{\partial f}{\partial q_i} dq = 0$$

We can combine these two relations to obtain

$$\int \sum_{i=1}^{s} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} + \sum_l \lambda_l \frac{\partial f}{\partial q_i} \right) dq_i = 0$$

Then Lagrange's Equations take the form

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = \sum_l \lambda_l \frac{\partial f_l}{\partial q_i}} \quad (*)$$

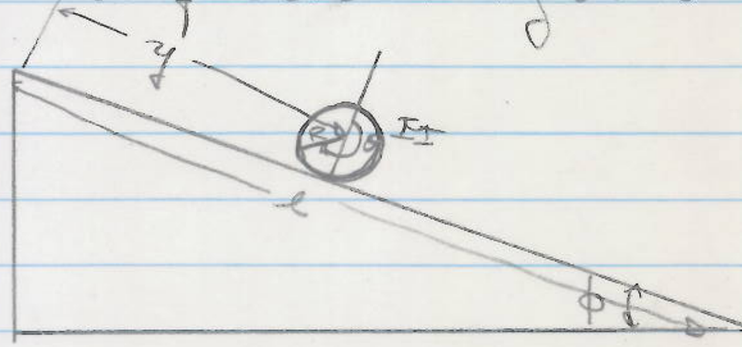
Here λ_l are the UNDETERMINED MULTIPLIERS and they represent the FORCES of CONSTRAINT

There are the same number λ_ℓ as the number of equations of constraint

The generalized forces of constraint Q_i are given by

$$Q_i = \sum_\ell \lambda_\ell \frac{\partial f_\ell}{\partial q_i}$$

Let us consider the case of a disk rolling down an inclined plane



$$T = \frac{1}{2} M \dot{y}^2 + \frac{1}{2} I \dot{\theta}^2$$

$$I = \frac{1}{2} M R^2$$

$$T = \frac{1}{2} M \dot{y}^2 + \frac{1}{4} M R^2 \dot{\theta}^2$$

$$U = Mg(\ell - y)\sin\phi$$ or potential energy is zero at the bottom of the ramp

$$L = T - U = \frac{1}{2} M \dot{y}^2 + \frac{1}{4} M R^2 \dot{\theta}^2 - Mg(\ell - y)\sin\phi = 0$$

The equation of the holonomic constraint giving the relation between the coordinates y and θ is

$$f(y, \theta) = y - R\theta = 0$$

Again: f = constraint function
λ = undetermined multiplier

$$f = y - R\theta \qquad \frac{\partial f}{\partial y} = 1 \qquad \frac{\partial f}{\partial \theta} = -R$$

Lagranges equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} - \lambda \frac{\partial f}{\partial y} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} - \lambda \frac{\partial f}{\partial \theta} = 0$$

Performing the operations on L and f, we obtain the equations of motion

$$M\ddot{y} - Mg\sin\phi - \lambda = 0 \qquad (1)$$

$$\frac{1}{2}MR^2\ddot{\theta} + \lambda R = 0 \qquad (2)$$

We have $y = R\theta \Rightarrow \ddot{\theta} = \frac{\ddot{y}}{R}$ (3)

substituting (3) into (2) yields

$$\lambda = -\frac{1}{2}M\ddot{y} \qquad (4)$$

Plugging (4) into (1) gives.

$$\ddot{y} = \frac{2g\sin\phi}{3}$$

which gives

$$\lambda = -\frac{1}{3} M g \sin\phi \qquad (5)$$

so that equ (2) becomes

$$\ddot{\theta} = \frac{2 g \sin\phi}{3R}$$

Thus we have three equations for the quantities $\ddot{y}$, $\ddot{\theta}$, and λ

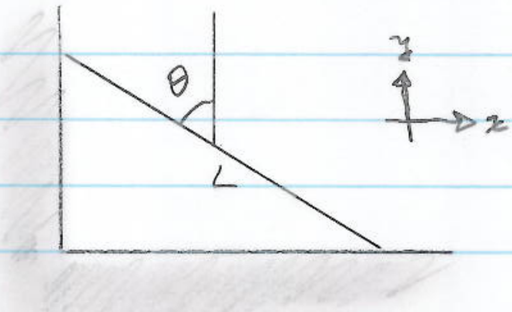
We note that if the disk were to slide without friction down the inclined plane the acceleration $\ddot{y}$ would be equal to $\ddot{y} = g\sin\phi$. The rolling constraint reduces the acceleration to $2/3$ of the value of frictionless sliding

The generalized forces of constraint are

$$Q_y = \lambda \frac{\partial f}{\partial y} = \lambda = -\frac{1}{3} M g \sin\phi \leftarrow \text{force of friction}$$

$$Q_\theta = \lambda \frac{\partial f}{\partial \theta} = -\lambda R = \frac{1}{3} M g R \sin\phi \leftarrow \text{torque.}$$

Consider the case of a ladder of length L that is inclined against a frictionless wall and floor as shown below. Find the equations of motion.



The position of the center of mass of the ladder, and its orientation can be described with the variables x, y, and θ.

We have the two constraining functions

$$f_1 = x - \frac{L}{2}\sin\theta \qquad f_2 = y - \frac{L}{2}\cos\theta$$

$$\frac{\partial f_1}{\partial x} = 1 \quad \frac{\partial f_1}{\partial y} = 0 \quad \frac{\partial f_1}{\partial \theta} = -\frac{L}{2}\cos\theta$$

$$\frac{\partial f_2}{\partial x} = 0 \quad \frac{\partial f_2}{\partial y} = 1 \quad \frac{\partial f_2}{\partial \theta} = +\frac{L}{2}\sin\theta$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 \; ; \quad I = \frac{1}{12}mL^2$$

$$U = mgy$$

$$\Rightarrow L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{mL^2}{24}\dot{\theta}^2 - mgy$$

LAGRANGE'S EQUATIONS:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum_l \lambda_l \frac{\partial f_l}{\partial q_i} \longleftarrow$$

7-29

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \lambda_1 \frac{\partial f_1}{\partial x} + \lambda_2 \frac{\partial f_2}{\partial x} \qquad (1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = \lambda_1 \frac{\partial f_1}{\partial y} + \lambda_2 \frac{\partial f_2}{\partial y} \qquad (2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \lambda_1 \frac{\partial f_1}{\partial \theta} + \lambda_2 \frac{\partial f_2}{\partial \theta} \qquad (3)$$

equ (1) → $m\ddot{x} = \lambda_1 \qquad (4)$

equ (2) → $m\ddot{y} + mg = \lambda_2 \qquad (5)$

eq (3) → $\frac{mL^2}{12}\ddot{\theta} = -\lambda_1\left(\frac{L}{2}\cos\theta\right) + \lambda_2\left(\frac{L}{2}\sin\theta\right) \qquad (6)$

Now $x = \frac{L}{2}\sin\theta$ | $y = \frac{L}{2}\cos\theta$

$\dot{x} = \frac{L}{2}\dot{\theta}\cos\theta$ | $\dot{y} = -\frac{L}{2}\dot{\theta}\sin\theta$

$\ddot{x} = \frac{L}{2}\ddot{\theta}\cos\theta - \frac{L}{2}\dot{\theta}^2\sin\theta$ | $\ddot{y} = -\frac{L}{2}\ddot{\theta}\sin\theta - \frac{L}{2}\dot{\theta}^2\cos\theta$

This means.

$$\lambda_1 = m\left(\frac{L}{2}\ddot{\theta}\cos\theta - \frac{L}{2}\dot{\theta}^2\sin\theta\right) \qquad (7)$$

$$\lambda_2 = m\left(-\frac{L}{2}\ddot{\theta}\sin\theta - \frac{L}{2}\dot{\theta}^2\cos\theta + g\right) \qquad (8)$$

Inserting (7) and (8) into (6)

$$\frac{mL^2}{12}\ddot{\theta} = -m\left(\frac{L}{2}\ddot{\theta}\cos\theta - \frac{L}{2}\dot{\theta}^2\sin\theta\right)\left(\frac{L}{2}\cos\theta\right) + m\left(-\frac{L}{2}\ddot{\theta}\sin\theta - \frac{L}{2}\dot{\theta}^2\cos\theta + g\right)\left(\frac{L}{2}\sin\theta\right)$$

7-30

$$\frac{\ddot{\theta}}{3} = \cancel{\dot{\theta}^2 \sin\theta\cos\theta} - \ddot{\theta}\cos^2\theta - \ddot{\theta}\sin^2\theta - \cancel{\dot{\theta}^2 \sin\theta\cos\theta} + \frac{2g}{L}\sin\theta$$

$$= -\ddot{\theta}(\cos^2\theta + \sin^2\theta) + 2\frac{g}{L}\sin\theta$$

$$\frac{4}{3}\ddot{\theta} = 2\frac{g}{L}\sin\theta \Rightarrow \boxed{\ddot{\theta} = \frac{3}{2}\frac{g}{L}\sin\theta}$$

Now what is the torque?

$$Q_\theta = \lambda_1 \frac{\partial f_1}{\partial \theta} + \lambda_2 \frac{\partial f_2}{\partial \theta}$$

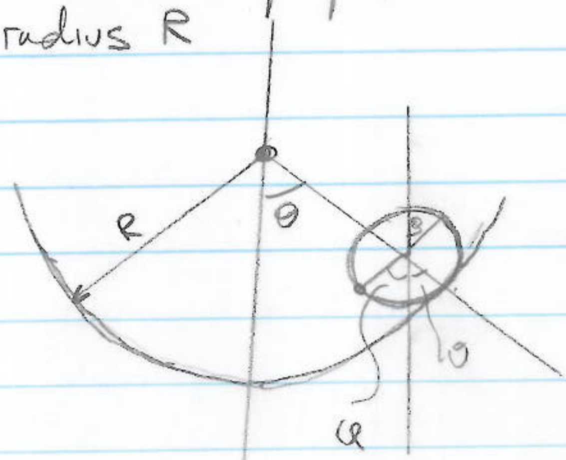
$$= \left(m\frac{L}{2}\ddot{\theta}\cos\theta - m\frac{L}{2}\dot{\theta}^2\sin\theta\right)\left(-\frac{L}{2}\cos\theta\right)$$
$$\left(-\frac{mL}{2}\ddot{\theta}\sin\theta - \frac{mL}{2}\dot{\theta}^2\cos\theta + mg\right)\left(\frac{L}{2}\sin\theta\right)$$

$$= -m\frac{L^2}{4}\ddot{\theta}\cos^2\theta + \cancel{m\frac{L^2}{4}\dot{\theta}^2\sin\theta\cos\theta} - \frac{mL^2}{4}\ddot{\theta}\sin^2\theta$$
$$- \cancel{\frac{mL^2}{4}\dot{\theta}^2\cos\theta\sin\theta} + mg\frac{L}{2}\sin\theta$$

$$\boxed{Q_\theta = \frac{1}{2}mgL\sin\theta - \frac{1}{4}mL^2\ddot{\theta}}$$

Example A sphere of radius ρ is constrained to roll without slipping on the lower half of the inner surface of a hollow cylinder of radius R

We shall take as our generalized coordinates θ and φ



7.31

$$T = \frac{1}{2}m\left[(R-\rho)\dot{\theta}\right]^2 + \frac{1}{2}I\dot{\varphi}^2$$

$$U = mg\left[R-(R-\rho)\cos\theta\right]$$

Recall that $I = \frac{2}{5}m\rho^2$

$$L = T - U = \frac{1}{2}m\left[(R-\rho)\dot{\theta}\right]^2 + \frac{1}{5}m\rho^2\dot{\varphi}^2 - mg\left[R-(R-\rho)\cos\theta\right] = 0$$

The equation of constraint:

$$f(\theta,\varphi) = \left[(R-\rho)\theta - \rho\phi\right] = 0$$

$\Rightarrow$ The frequency of oscillation is

$$\omega = \left[\frac{5g}{7(R-\rho)}\right]^{1/2}$$

Hamiltonian FUNCTION: CONSERVATION LAWS and symmetry principles

A system that does not interact with anything outside the system. is called A CLOSED SYSTEM! For such a system there are always SEVEN CONSTANTS or INTEGRALS of MOTION

- The linear momentum (3 components)
- The angular momentum (3 components)
- The total energy.

Conservation of Linear Momentum

Let us consider a Lagrangian of a closed system in an inertial reference frame. Because space is homogeneous in an inertial frame, the Lagrangian of a closed system is unaffected by a translation of the entire system in space.

For the sake of simplicity, let us consider a single particle with a Lagrangian $L(q, \dot{q})$. The variation in L due to the variation in the generalized coordinates ($\delta\vec{q} = \sum_i \delta q_i \hat{q}_i$) is

$$\delta L = \sum_i \frac{\partial L}{\partial q_i}\delta q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i = 0$$

Since we are only varying the displacements, our δq_i are not functions of time (explicitly or implicitly)

$$\delta \dot{q}_i = \delta\left(\frac{dq_i}{dt}\right) = \frac{d}{dt}(\delta q_i) = 0$$

Therefore δL becomes

$$\delta L = \sum_i \frac{\partial L}{\partial q_i} \delta q_i = 0$$

Because each of the δq_i is an INDEPENDENT displacement, δL will vanish identically if and only if each of the partial derivatives of L vanishes

$$\frac{\partial L}{\partial q_i} = 0$$

Thus Lagrange's Equations reduce to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_i} = \text{constant} = \frac{\partial}{\partial \dot{q}_i}(T - U) = \frac{\partial}{\partial \dot{q}_i}(T)$$

$$= \frac{\partial}{\partial \dot{q}_i}\left(\frac{1}{2} m \sum_i \dot{q}_i^2\right) = m\dot{q}_i$$

$$= p_i = \text{constant}$$

Since the motion of a single particle can be described by three generalized components of the linear momentum in Euclidean space, there will be three constants (e.g. p_x, p_y, and p_z) of motion if

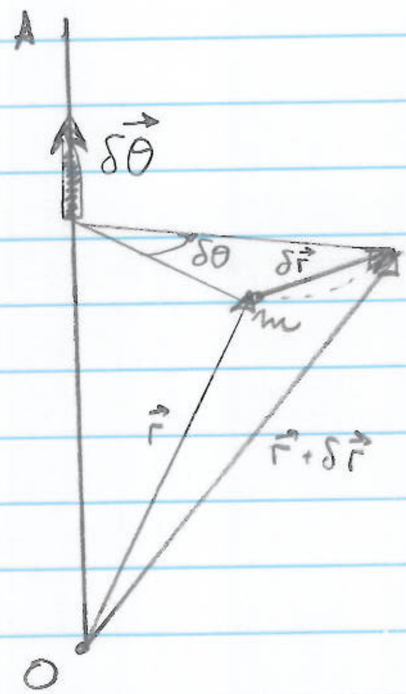
⇒ If the Lagrangian of a system, closed or otherwise, is invariant with respect to a translation in a certain direction, then the linear momentum of the system in that direction is constant in time.

[This implies that if space is homogeneous, then the linear momentum $\vec{p}$ of a closed system is constant in time]

CONSERVATION OF ANGULAR MOMENTUM

Another outstanding property of an inertial system is that SPACE IS ISOTROPIC IN AN INERTIAL FRAME; that is, a closed system is unaffected by the orientation or rotation of the entire system. This implies that the Lagrangian of a closed system remains invariant if the system is rotated through an infinitesimal angle.

7-35



Rotation about an axis OA of a point particle m at a distance $\vec{r}$ from the origin O

Let us consider a system consisting of a single particle. The change in the Lagrangian is

$$\delta L = \sum_i \frac{\partial L}{\partial q_i}\delta q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i = 0$$

By definition (in homogeneous space)

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Then from Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$$\Rightarrow \dot{p}_i = \frac{\partial L}{\partial q_i}$$

we then have

$$\delta L = \sum_i \dot{p}_i \delta q_i + \sum_i p_i \delta \dot{q}_i$$

From our figure above, we have a point particle located a distance $\vec{r}$ from the origin O. The system is rotated through an angle $\delta\theta$ about an axis. The value $\vec{r}$ changes so that

$$\delta\vec{r} = \delta\vec{\theta} \times \vec{r}$$

← You can verify this by means of the Right Hand Rule.

This leads to a change in the velocity:

$$\delta\dot{\vec{r}} = \delta\vec{\theta} \times \dot{\vec{r}}$$

← The axis is by construction FIXED.

We now set $p_i \equiv \vec{p}$ and $\dot{q}_i \equiv \dot{\vec{r}}$, we get for the three components of the vector $(i=1,2,3)$

$$\delta L = \dot{\vec{p}} \cdot \delta\vec{r} + \vec{p} \cdot \delta\dot{\vec{r}} = 0$$

From the above

$$\delta L = \dot{\vec{p}} \cdot (\delta\vec{\theta} \times \vec{r}) + \vec{p} \cdot (\delta\vec{\theta} \times \dot{\vec{r}}) = 0$$

We may permute in cyclic order the factors of the triple product without changing its value. (see Ch. 1)

$$\delta L = \delta\vec{\theta} \cdot (\vec{r} \times \dot{\vec{p}}) + \delta\vec{\theta} \cdot (\dot{\vec{r}} \times p) = 0$$

7-37

$$\delta L = \delta\vec{\theta} \cdot [(\vec{r} \times \dot{\vec{p}}) + (\dot{\vec{r}} \times \vec{p})] = 0$$

$$= \delta\vec{\theta} \cdot \left(\frac{d}{dt}(\vec{r} \times \vec{p})\right) = 0$$

But $\vec{J}$ is the angular momentum about the given axis. Therefore

$$\delta\vec{\theta} \cdot \frac{d\vec{J}}{dt} = 0$$

Since $\delta\vec{\theta}$ is arbitrary, we must have:

$$\frac{d\vec{J}}{dt} = 0$$

That is,

$$\vec{J} = \vec{r} \times \vec{p} = \text{constant}$$

Here $\vec{J}$ has three components.

If the Lagragian remains invariant under rotation about an axis, then the angular momentum of the system about this axis will remain constant in time.

Conservation of Energy and the HAMILTONIAN

An outstanding property of an inertial frame is that TIME IS HOMOGENEOUS WITHIN AN INERTIAL FRAME OF REFERENCE. This implies that the Lagrangian of a closed system cannot be an explicit function of time

The total differential of L

$$dL = \sum_i \frac{\partial L}{\partial q_i}\partial q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\partial \dot{q}_i + \frac{\partial L}{\partial t}\partial t$$

But we have $\frac{\partial L}{\partial t} = 0$

The total time derivative of L reduces to

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i}\frac{dq_i}{dt} + \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial t}$$

or

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$$

From Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

Substituting for $\frac{\partial L}{\partial q_i}$ above we obtain

$$\frac{dL}{dt} = \sum_i \dot{q}_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \sum_i \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i}$$

$$\frac{dL}{dt} = \sum_i \frac{d}{dt}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right)$$

or

$$\frac{d}{dt}\left[\sum_i \left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) - L\right] = 0$$

The quantity in brackets is therefore constant in time. We shall denote this constant by H, called the Hamiltonian H.

$$H = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = \text{const.}$$

[But we define $p_x = \frac{\partial T}{\partial \dot{x}}$ and we set $U = U(x)$ ← Not a function of velocity.

$$H = \sum_i \dot{q}_i p_i - L = \text{constant}$$]

Hence H is a constant of the motion if L is not an explicit function of time t, i.e., $\frac{\partial L}{\partial t} = 0$

⟹ We observe that the <u>Hamiltonian</u> assumes a special form if the following two conditions are met:

① The potential energy U is independent of the velocity coordinate so that

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial (T-U)}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} \; ; \; U = U(q)$$

(2) The equations representing the transformation of coordinates do not contain time explicitly. This ensures that the kinetic energy is a homogeneous quadratic function of the $\dot{q}_i$

Now according to Euler's Theorem, which states that $f(y_k)$ is a homogeneous function of the y_k that is of degree n $(f = q_i^n)$, then

$$\sum_k y_k \frac{\partial f}{\partial y_k} = nf$$ ← see section 7.8

For example, if the kinetic energy T is a homogeneous quadratic function

$$T \Rightarrow f = \frac{1}{2} m (\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2)$$

$$\dot{q}_i \Rightarrow y_k$$

$$\sum_k y_k \frac{\partial f}{\partial y_k} \rightarrow \sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = \sum_i \dot{q}_i (m\dot{q}_i)$$

$$= 2T$$

We have then

$$H = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = \text{constant}$$

$$= 2T - (T-U) = T+U = E = \text{const.}$$

Here E is the total energy and is constant.

Recall that $H = E$ iff

- $L \neq L(t)$
- $U = U(q)$ & $U \neq U(\dot{q})$
- T is a homogeneous quadratic function.

Consider a conservative system and let the description be made in terms of generalized coordinates in motion wrt a fixed, rectangular axes. The transformation equations then contain the time EXPLICITLY, and the kinetic energy is NOT a homogeneous quadratic function of the generalized velocities. The choice of a mathematically convenient set of generalized coordinates cannot alter the physical fact that energy is always conserved but

$$H \neq E \;!$$

SYMMETRY PROPERTIES AND CONSERVATION LAWS

Property in an INERTIAL FRAME	Conserved Quantity	Property of the LAGRANGIAN
Homogeneous Space	LINEAR MOMENTUM	Invariant to translation
ISOTROPIC SPACE	ANGULAR MOMENTUM	Invariant to rotation.
HOMOGENEOUS TIME	TOTAL ENERGY	Not an EXPLICIT function of TIME.

Canonical Equations of Motion (Hamiltonian Dynamics)

We found earlier that if $U \neq U(\dot{x})$ $(U = U(x))$ then

$$p_x = \frac{\partial L}{\partial \dot{x}}$$

By way of analogy in which the Lagrangian is expressed in generalized coordinates, we can define the GENERALIZED MOMENTA:

$$\boxed{p_i \equiv \frac{\partial L}{\partial \dot{q}_i}}$$

N.B. $[p_i]$ Need not have the units of linear momentum!

7-43

Now $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

$$\frac{d}{dt}(p_i) = \frac{\partial L}{\partial q_i}$$

$$\boxed{\dot{p}_i = \frac{\partial L}{\partial q_i}}$$

Now

$$dL = \sum_i \frac{\partial L}{\partial q_i} dq_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt$$

$$dL = \sum_i \dot{p}_i \, dq_i + \sum_i p_i \, d\dot{q}_i + \frac{\partial L}{\partial t} dt$$

We observe that

$$d\left(\sum_i \dot{q}_i p_i\right) = \sum_i p_i \, d\dot{q}_i + \sum \dot{q}_i \, dp_i$$

$$\Rightarrow \sum_i p_i \, d\dot{q}_i = d\left(\sum_i \dot{q}_i p_i\right) - \sum \dot{q}_i \, dp_i$$

Inserting this result into our equation for dL we get

$$dL = \sum_i \dot{p}_i \, dq_i + d\left(\sum \dot{q}_i p_i\right) - \sum \dot{q}_i \, dp_i + \frac{\partial L}{\partial t} dt$$

$$d\left(\sum_i \dot{q}_i p_i - L\right) = \sum \dot{q}_i dp_i - \sum \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt$$

Now we found earlier that

$$H = \sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = \text{const}$$

and that $\frac{\partial L}{\partial \dot{q}_i} = p_i$

$$\Rightarrow dH = \sum (\dot{q}_i dp_i - \dot{p}_i dq_i) - \frac{\partial L}{\partial t} dt$$

Let us examine this equation.

$$\Rightarrow dH = \sum_i \frac{\partial H}{\partial p_i} dp_i + \sum_i \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial t} dt$$

Here $H = H(q_i, p_i, t)$

[And $L = L(q_i, \dot{q}_i, t)$]

If we equate the coefficients for these expansions in dH we see

$$\boxed{\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned}}$$

$\leftarrow$ Hamilton's Equations of motion.

Generalized Momenta and Cyclic (or IGNORABLE) COORDINATES.

The Lagrangian L is described in terms of the generalized coordinates q_k and the generalized velocities $\dot{q}_k$. Furthermore if a Lagragiam is an explicit function of time, we may write:

$$L = L(q, \dot{q}; t)$$

And the Lagrange formalism leads to n second-order differential equations.

An alternative to the Lagrange formalism is the Hamilton formalism. Hamilton's formalism is carried out in terms of the generalized coordinates and generalized momenta

$$H = H(q, p; t)$$

Such formalism for a system of n degrees of freedom leads to $2n$ first-order differential equations. These $2n$ first-order differential equations are much easier to solve than n second-order differential equation, as in the case for Lagrange's formalism

To begin, let us define the generalized momentum (again). As a simple example, let us consider the motion of a single particle moving with velocity $\dot{x}$ along the x-axis. The kinetic energy of such a particle is

$$T = \frac{1}{2} m \dot{x}^2$$

$$\Rightarrow \quad p_x = \frac{\partial T}{\partial \dot{x}} = m\dot{x}$$

Furthermore, if $U \neq U(\dot{x})$, then the momentum p_x may also be written as

$$p_x = \frac{\partial L}{\partial \dot{x}}$$

IN GENERAL (with generalized coordinates) we define

$$p_k = \frac{\partial L}{\partial \dot{q}_k} \quad \leftarrow \text{This is the generalized momentum}$$

The generalized momentum p_k is called the

CONJUGATE MOMENTUM p_k

which is conjugate to the generalized coordinate q_k

$\Rightarrow$ N.B. The conjugate momentum p_k NEED NOT have units of mass x velocity !

For a conservative system, Lagrange's equations are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

$$\dot{p}_k = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \qquad \left(\text{because } p_k = \frac{\partial L}{\partial \dot{q}_k}\right)$$

Therefore

$$\dot{p}_k = \frac{\partial L}{\partial q_k}$$

Let us now explore the connection between the generalized momenta and the constants of motion.

⇒ Quanties that are functions of the coordinates and/or velocities that remain constant in time are called

CONSTANTS OF MOTION ⇐

Suppose in the expression for the Lagrangian L of a system a certain coordinate q_α does not occur explicitly. Then

$$\frac{\partial L}{\partial q_\alpha} = 0$$

We have then

$$\dot{p}_\alpha = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = 0$$

which upon integration yields

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} = \text{constant} = C_\alpha$$

That is to say that whenever a Lagrangian function does not contain a coordinate q_α explicitly, the corresponding generalized momentum p_α is a CONSTANT OF THE MOTION. The coordinate q_α is said to be <u>cyclic</u> or <u>ignorable</u>

For example, let us consider the motion of a particle in a central force field

$$L = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$

Since $L \neq L(\theta)$, the coordinate θ is cyclic (or ignorable), and the generalized momentum corresponding to θ is

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{constant}$$

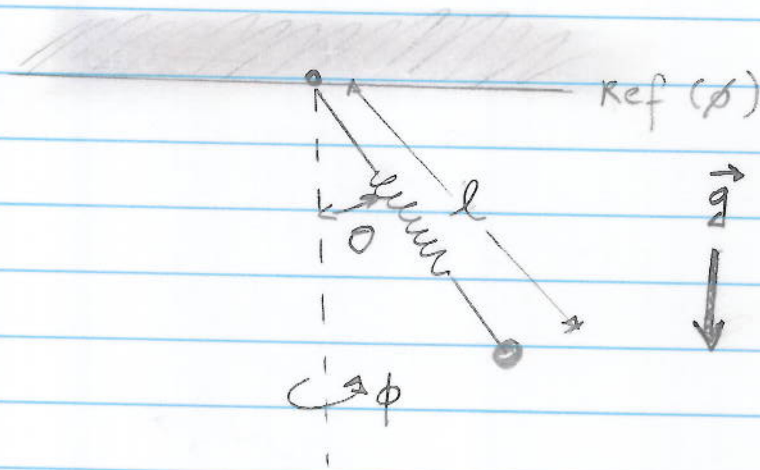
Here p_θ is the angular momentum and is a constant of the motion

Example Using the Hamiltonian method, find the equations of motion for a spherical pendulum of mass m and unstretched length l_0 and spring constant k.

$$T = \frac{1}{2}m\left(\dot{l}^2 + l_0^2\dot{\theta}^2 + l_0^2\sin^2\theta\,\dot{\phi}^2\right)$$

$$U = -mgl_0\cos\theta + \frac{1}{2}k\Delta l^2 \quad ; \quad \Delta l = l - l_0$$

The generalized momenta:



$$p_l = \frac{\partial L}{\partial \dot{l}} = m\dot{l}$$

$$p_\theta = ml_0^2\dot{\theta}$$

$$p_\phi = ml_0^2\sin^2\theta\,\dot{\phi}$$

Using the above equations we can solve for $\dot{l}$, $\dot{\theta}$, and $\dot{\phi}$ in terms of p_l, p_θ, and p_ϕ

Since $L \neq L(t)$ and $U \neq U(\dot{q})$ we can determine the Hamiltonian from $H = T + U$

$$\dot{l} = \frac{p_l}{m}, \quad \dot{\theta} = \frac{p_\theta}{ml_0^2}, \quad \dot{\phi} = \frac{p_\phi}{ml_0^2\sin^2\theta}$$

$$H = T + U = \frac{1}{2m}\left(p_l^2 + \frac{p_\theta^2}{l_0^2} + \frac{p_\phi^2}{l_0^2\sin^2\theta}\right)$$

$$- mgl_0\cos\theta + \frac{1}{2}k(l - l_0)^2$$

The equations of motion:

$$\dot{l} = \frac{\partial H}{\partial p_l} = \frac{p_l}{m}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m l_0^2}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{m l_0^2 \sin^2\theta}$$

$$\dot{p}_l = -\frac{\partial H}{\partial l} = -2k(l - l_0)$$

Note that the units are Newtons $\Rightarrow F_l = \frac{dp_l}{dt}$ (Newton's second law)

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2 \cos\theta}{m l_0^2 \sin^3\theta} - mg\, l_0 \sin\theta$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \leftarrow !$$

Because ϕ is cyclic (or ignorable), the momentum p_ϕ about the symmetry axis is constant. (i.e. Angular momentum is conserved)

<u>Example</u> A particle of mass m is attracted to a force center with a force $f = -Ar^{\alpha-1}$ directed radially towards this force center. ($\alpha > 0$) Find Hamilton's equations of motion

The force is conservative because we can find a scalar function U so that $-\nabla U = \vec{f}$

$$U = \frac{A}{\alpha} r^\alpha$$

7-51

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$L = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{A}{\alpha}r^\alpha$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \Rightarrow \dot{r} = \frac{p_r}{m}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2}$$

SINCE $L \neq L(t)$ (explicitly) and $U \neq U(\dot{q})$, the Hamiltonian is the total mechanical energy

$$H = T + U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{A}{\alpha}r^\alpha$$

$$= \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2}\right] + \frac{A}{\alpha}r^\alpha$$

Hamilton's equations of motion are

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \qquad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - Ar^{\alpha-1}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \quad \text{as cyclic in } \theta!$$

Poisson Brackets

Consider any two continuous functions of the generalized coordinates and generalized momenta $g = g(q,p)$ and $h = h(q,p)$. We define the Poisson Brackets:

$$[g, h] = \sum_k \left(\frac{\partial g}{\partial q_k} \frac{\partial h}{\partial p_k} - \frac{\partial g}{\partial p_k} \frac{\partial h}{\partial q_k} \right)$$

→ The formalism of POISSON BRACKETS is of utmost importance in QM.

By the definition of the total derivative

$$\frac{dg}{dt} = \frac{\partial g}{\partial t} + \sum_k \left(\frac{\partial g}{\partial q_k} \dot{q}_k + \frac{\partial g}{\partial p_k} \dot{p}_k \right)$$

From the canonical equations

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \qquad \dot{p}_k = - \frac{\partial H}{\partial q_k}$$

Hence

$$\frac{dg}{dt} = \frac{\partial g}{\partial t} + \sum_k \left(\frac{\partial g}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial g}{\partial \dot{q}_k} \frac{\partial H}{\partial q_k} \right)$$

$$= \frac{\partial g}{\partial t} + [g, H]$$

$$\dot{q}_j = \frac{\partial H}{\partial p_j}$$

$$[q_j, H] = \sum_k \left(\frac{\partial q_j}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial q_j}{\partial p_k}\frac{\partial H}{\partial q_k} \right)$$

However $\frac{\partial q_j}{\partial q_k} = \delta_{jk}$ and $\frac{\partial q_j}{\partial p_k} = 0$ for all j and k

Hence

$$[q_j, H] = \frac{\partial H}{\partial p_j} = \dot{q}_j .$$

If the Poisson Bracket of two quanties vanishes, the quantities are said to commute. If a quantity DOES NOT depend EXPLICITLY on the time, it will commute with the Hamiltonian. And hence is a constant of the motion of the system.

$$[g, H] = 0$$

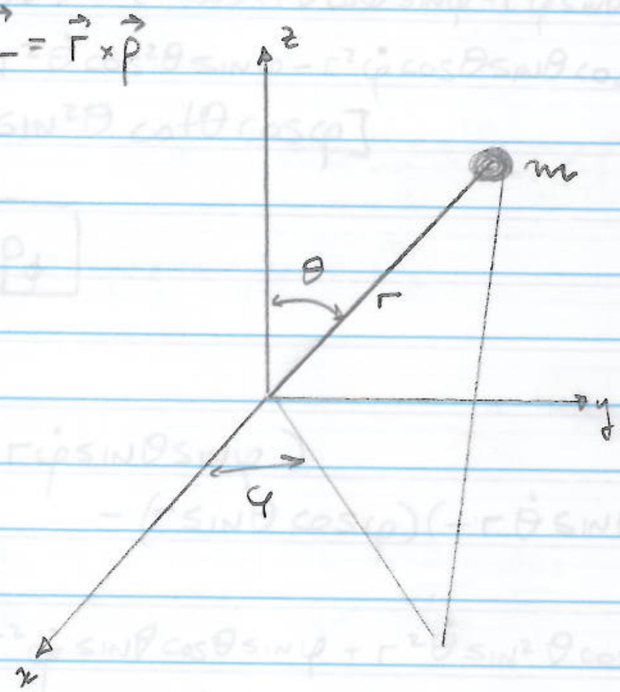
$$\frac{dg}{dt} = \frac{\partial g}{\partial t} + [g, H] \; ; \quad \frac{\partial g}{\partial t} = 0 \text{ and } [g, H] = 0$$

$$\Rightarrow \frac{dg}{dt} = 0 \Rightarrow g = \text{const.}$$

Let us revisit the spherical pendulum in Hamiltonian formulation, using spherical polar coordinates for the q_i. We wish to evaluate directly in terms of the canonical variables the following Poisson Brackets:

$$[L_x, L_y] \quad , \quad [L_y, L_z] \quad , \quad [L_z, L_x]$$

$L \equiv$ Angular momentum $\quad \vec{L} = \vec{r} \times \vec{p}$



We found that

$$p_\theta = mr^2\dot{\theta}$$
$$p_\phi = mr^2 \sin^2\theta \, \dot{\varphi}$$

$$L_x = y p_z - z p_y$$
$$= m(y\dot{z} - z\dot{y})$$

$$L_y = z p_x - x p_z$$
$$= m(z\dot{x} - x\dot{z})$$

$$L_z = x p_y - y p_x$$
$$= m(x\dot{y} - y\dot{x})$$

$$x = r\sin\theta\cos\varphi$$
$$\dot{x} = r\dot{\theta}\cos\theta\cos\varphi - r\dot{\varphi}\sin\theta\sin\varphi$$

$$y = r\sin\theta\sin\varphi$$
$$\dot{y} = r\dot{\theta}\cos\theta\sin\varphi + r\dot{\varphi}\sin\theta\cos\varphi$$

$$z = r\cos\theta$$
$$\dot{z} = -r\sin\theta$$