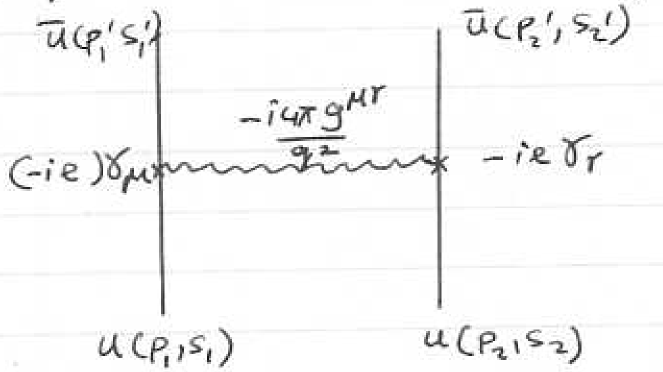
⟨e-e (Møller) Scattering at first order⟩

↳ Batch $\overline{XIV}$

We will now calculate the differential cross-section for the scattering of two identical Fermions, ie electrons. Let us assume that electron 1 and electron 2 scatter off each other with initial momentum and spin variables : $(p_1,s_1)(p_2,s_2)$ and final momentum and spin variables (p_1',s_1') (p_2',s_2')

Naively one might sketch the following graph to describe this process; obtaining:



(This is a Feynman graph).

[Note: the $g^{\mu\nu}$ on the virtual photon line ensures that the index of the $\gamma_\nu \Rightarrow \gamma_\mu$ so that the matrix element is a scalar...]

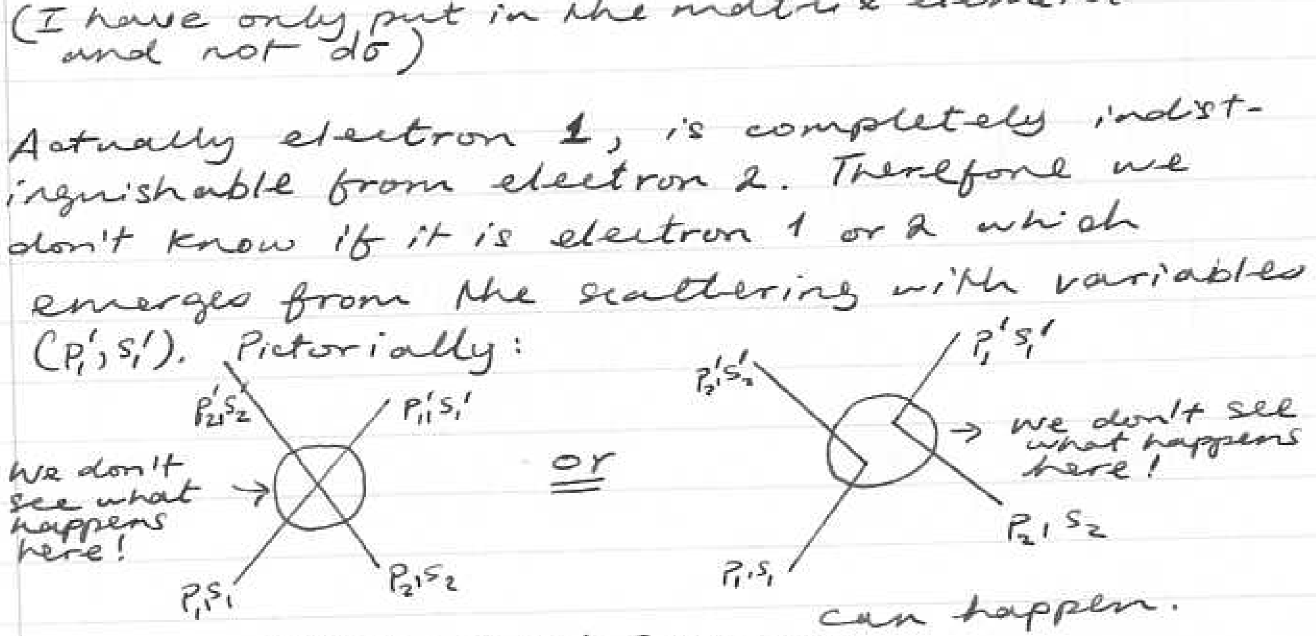
$$\hookrightarrow \bar{u}(p_1',s_1')(-ie)\gamma_\mu u(p_1,s_1)\left(\frac{-i4\pi g^{\mu\nu}}{q^2}\right)\bar{u}(p_2',s_2')(-ie\gamma_\nu)u(p_2,s_2)$$

$$= ie^2\bar{u}(p_1',s_1')\gamma_\mu u(p_1,s_1)\frac{4\pi}{q^2}\bar{u}(p_2',s_2')\gamma^\mu u(p_2,s_2)$$

↳ as our matrix element.

(I have only put in the matrix element and not $d\sigma$)

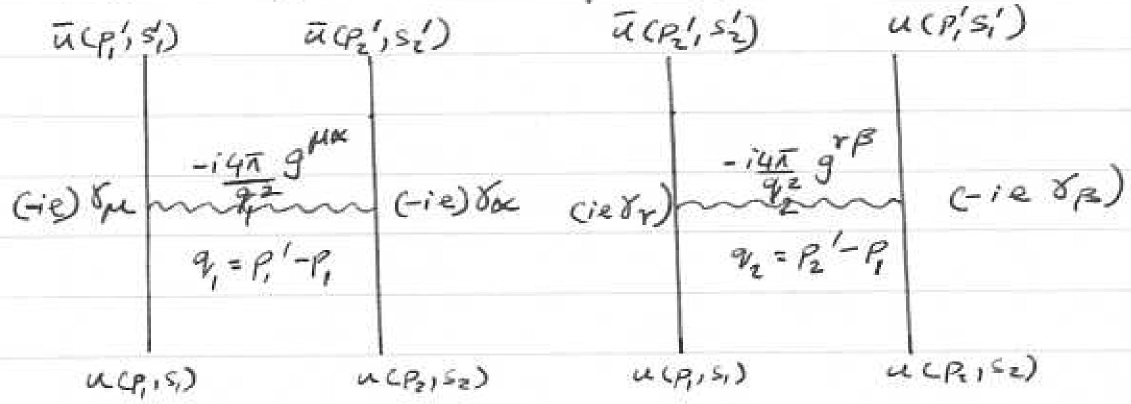
Actually electron **1**, is completely indistinguishable from electron 2. Therefore we don't know if it is electron 1 or 2 which emerges from the scattering with variables (p_1',s_1'). Pictorially:



can happen.

(These are not Feynmann graphs!)

We need to consider <u>two</u> Feynmann graphs contributing to this process.



Note that since we have swapped the indentities of two identical Fermions, we must have the contributions of the two graphs with a relative <u>negative</u> sign:

$$M_{fi} =$$

$$(-ie)(-i)(-ie)\Big[\bar{u}(p_1',s_1')\gamma_\mu u(p_1,s_1)\frac{4\pi}{q_1^2}\bar{u}(p_2',s_2')\gamma^\mu u(p_2,s_2) - \bar{u}(p_2's_2')\gamma_\gamma u(p_1,s_1)\frac{4\pi}{q_2^2}\bar{u}(p_1',s_1')\gamma^\gamma u(p_2 s_2)\Big]$$

with

$$d\sigma = \frac{m m}{\sqrt{(p_1\cdot p_2)^2 - (m^2)^2}} \times |M_{fi}|^2 (2\pi)^4 \delta(p_2' + p_1' - p_2 - p_1) \times \frac{m}{(2\pi)^3} \times \frac{m}{(2\pi)^3}\frac{d^3p_1'}{E_1'}\frac{d^3p_2'}{E_2'} \quad \text{--- (1)}$$

as the expression for the cross-section. We'll return to expression (1) in a minute.

$$\frac{1}{4}\sum_{s_f,s_i}|M_{fi}|^2 = \frac{1}{4}\sum_{s_f,s_i} e^4\Big[|\bar{u}(p_1's_1')\gamma_\mu u(p_1,s_1)\frac{4\pi}{q_1^2}\bar{u}(p_2's_2')\gamma^\mu u(p_2,s_2)|^2$$

$$+ |\bar{u}(p_2's_2')\gamma_\gamma u(p_1,s_1)\frac{4\pi}{q_2^2}\bar{u}(p_1's_1')\gamma^\gamma u(p_2,s_2)|^2$$

$$- \bar{u}(p_1',s_1')\gamma_\mu u(p_1,s_1)\frac{4\pi}{q_1^2}\bar{u}(p_2',s_2')\gamma^\mu u(p_2,s_2)\Big[\bar{u}(p_2',s_2')\gamma_\gamma u(p_1,s_1)$$

$$\times \frac{4\pi}{q_2^2}\bar{u}(p_1',s_1')\gamma^\gamma u(p_2,s_2)\Big]^* - \bar{u}(p_2's_2')\gamma_\gamma u(p_1,s_1)\frac{4\pi}{q_2^2}\bar{u}(p_1',s_1')\gamma^\gamma u(p_2,s_2)$$

$$\times\Big[\bar{u}(p_1',s_1')\gamma_\mu u(p_1,s_1)\frac{4\pi}{q_1^2}\bar{u}(p_2's_2')\gamma^\mu u(p_2,s_2)\Big]^*$$

$$\Big]$$

closing brackets.

Let me put in the spin indices correctly on the summation sign:

$$\frac{1}{4}\sum_{\text{Spins}}|M_{fi}|^2 = \frac{e^4}{4}\sum_{s_1 s_1' s_2 s_2'}\Big\{\Big|\bar{u}(p_1', s_1)\gamma_\mu u(p_1, s_1)\frac{4\pi}{q_1^2}\bar{u}(p_2', s_2')\gamma^\mu u(p_2, s_2)\Big|^2$$

$$+\Big|\bar{u}(p_2', s_2')\gamma_r u(p_1, s_1)\frac{4\pi}{q_2^2}\bar{u}(p_1', s_1')\gamma^r u(p_2, s_2)\Big|^2$$

$$-\bar{u}(p_1' s_1')\gamma_\mu u(p_1, s_1)\frac{4\pi}{q_1^2}\bar{u}(p_2', s_2')\gamma^\mu u(p_2, s_2)\Big[\bar{u}(p_2' s_2')\gamma_r u(p_1, s_1)$$

$$\frac{4\pi}{q_2^2}\bar{u}(p_1', s_1')\gamma^r u(p_2, s_2)\Big]^* - \bar{u}(p_2' s_2')\gamma_r u(p_1, s_1)\frac{4\pi}{q_2^2}\bar{u}(p_1', s_1')$$

$$\gamma^r u(p_2, s_2)\Big[\bar{u}(p_1' s_1')\gamma_\mu u(p_1 s_1)\frac{4\pi}{q_1^2}\bar{u}(p_2', s_2')\gamma^\mu u(p_2, s_2)\Big]^*\Big\}$$

The last two terms are cross-terms from the two graphs.

The first two terms are squares, we've already done these before in electron proton scattering, if you look in batch $\overline{XI}$ page 8 at the top the first $|\quad|^2$ term is

$$\frac{1}{2}Tr\Big[\frac{\not p_1'+m}{2m}\gamma^\mu\frac{\not p_1+m}{2m}\gamma^r\Big]\times\frac{1}{2}Tr\Big[\frac{\not p_2'+m}{2m}\gamma_\mu\frac{(\not p_2+m)}{2m}\gamma_r\Big]\times\left(\frac{4\pi}{q_1^2}\right)^2$$

(this is from the expression on page 11 Batch $\underline{IX}$ with $p_i \to p_1$ and $P_f \to \underline{p_1}'$, $P_i \to P_2$ and $p_f \to P_2'$)

Thus the second $|\quad|^2$ term requires the substitution: $p_i \to p_1 \quad p_f \to P_2'$, $P_i \to P_2 \quad P_f \to P_2'$ and we get:

$$e^4\frac{1}{2}Tr\Big[\frac{\not p_2'+m}{2m}\gamma^\mu\frac{(\not p_1+m)}{2m}\gamma^r\Big]\frac{1}{2}Tr\Big[\frac{\not p_1'+m}{2m}\gamma_\mu\frac{(\not p_2+m)}{2m}\gamma_r\Big]\times\left(\frac{4\pi}{q_2^2}\right)^2$$

(This is just to remind you what the traces looked like; I might as well have jumped to the final expression, which is the scalar product at the bottom of page 12 (Batch $\underline{IX}$), the expression is in terms of p_f and P_f, p_i and P_i and M & m:

$$\frac{e^4}{2m^2M^2}\Big[(p_f\cdot P_f)(p_i\cdot P_i)+(p_f\cdot P_i)(p_i\cdot P_f) - p_f\cdot p_i M^2 - P_f\cdot P_i m^2 + 2m^2M^2\Big]$$

(its missing the e^4 due to different context but I've put that in here). $\quad$ L (2).

We need only use this expression (2), to evaluate the $| \ |^2$ terms we need to make two sets of substitutions :

$$P_i \to P_1 \quad P_i \to P_2 \quad \text{and } M \to m \qquad \text{for the first}$$
$$P_f \to P_1' \quad P_f \to P_2' \qquad\qquad | \ |^2 \text{ term.}$$

$$P_i \to P_1 \quad P_i \to P_2 \quad \text{and } M \to m \qquad \text{for the second}$$
$$P_f \to P_2' \quad P_f' \to P_2' \qquad\qquad | \ |^2 \text{ term.}$$

and, we shouldn't forg g the factors of $\left(\frac{4\pi}{q_1^2}\right)^2$ and $\left(\frac{4\pi}{q_2^2}\right)^2$

$$e^4\left(\frac{4\pi}{q_1^2}\right)^2 \frac{1}{2m^4}\Big\{(P_1'\cdot P_2')(P_1\cdot P_2) + (P_2'\cdot P_1)(P_1'\cdot P_2) - (P_1'\cdot P)m^2 - (P_2'\cdot P_2)m^2 + 2m^4\Big\} \to (3)$$

$$e^4\left(\frac{4\pi}{q_2^2}\right)^2 \frac{1}{2m^4}\Big\{(P_2'\cdot P_1')(P_1\cdot P_2) + (P_2'\cdot P_2)(P_1\cdot P_1') - (P_2'\cdot P_1)m^2 - (P_2\cdot P_1')m^2 + 2m^4\Big\} \to (4)$$

Recall that $\underline{q_1 = P_1 - P_1'}$ and $\underline{P_1 - P_2' = q_2}$.

Lets now calculate the cross term, or the interference term, where I have extracted the (-) & the factors of $\frac{4\pi}{q_i^2}$ $i=1,2$.

$$\left(-\frac{1}{4}\right)\frac{4\pi}{q_1^2}\frac{4\pi e^4}{q_2^2}\sum_{\substack{s_1 s_2\\ s_1' s_2'}}\Big\{\bar{u}(P_1,s_1)\gamma_\mu u(P_1,s_1)\,\bar{u}(P_2',s_2')\gamma^\mu u(P_2,s_2)\big[\bar{u}(P_2',s_2')\gamma_\gamma u(P_1,s_1)$$
$$\times\,\bar{u}(P_1',s_1')\gamma^\gamma u(P_2,s_2)\big]^* + \bar{u}(P_2',s_2')\gamma_\gamma u(P_1,s_1)\,\bar{u}(P_1',s_1')\gamma^\gamma u(P_2,s_2)$$
$$\times\big[\bar{u}(P_1',s_1')\gamma_\mu u(P_1,s_1)\,\bar{u}(P_2',s_2')\gamma^\mu u(P_2,s_2)\big]^*\Big\}$$

Recall we have prooved $(\bar{u}_f\gamma^\mu u_i)^* = \bar{u}_i\gamma^\mu u_f$
lets use this to transform all the complex conjugates in the above expression.

$$\Rightarrow \left(-\frac{1}{4}\right)\frac{4\pi}{q_1^2}\frac{4\pi}{q_2^2}\sum_{\substack{s_1 s_2\\ s_1' s_2'}} e^4\Big\{\bar{u}(P_1',s_1')\gamma_\mu u(P_1,s_1)\,\bar{u}(P_2',s_2')\gamma^\mu u(P_2,s_2)$$
$$\times\,\bar{u}(P_2,s_2)\gamma^\gamma u(P_1',s_1')\,\bar{u}(P_1,s_1)\gamma_\gamma u(P_2',s_2')$$
$$+\,\bar{u}(P_2',s_2')\gamma_\gamma u(P_1,s_1)\,\bar{u}(P_1',s_1')\gamma^\gamma u(P_2,s_2)\,\bar{u}(P_2,s_2)\gamma^\mu u(P_2',s_2')$$
$$\times\,\bar{u}(P_1,s_1)\gamma_\mu u(P_1',s_1')\Big\} \longrightarrow (5)$$

Lets look at the first in the sum of two terms:

$$\sum_{s_1 s_2 s_1' s_2'}\left\{\underline{\bar u(p_1',s_1')\gamma_\mu u(p_1,s_1)}\,\underline{\bar u(p_2',s_2')\gamma^\mu u(p_2,s_2)}\,\underline{\bar u(p_2,s_2)\gamma^\nu \times u(p_1',s_1')}\,\underline{\bar u(p_1,s_1)\gamma_\nu u(p_2',s_2')}\right\}$$

each of the underlined terms is summed over all the γ matrix elements so each underlined term can be moved about as a "block".

lets now begin the manipulation. For brevity's sake lets denote $\bar u(p_1',s_1')$ by $\bar u_1'$ etc:

$$\sum_{s_1}\sum_{s_2}\sum_{s_1'}\sum_{s_2'}\left\{\underline{\bar u_1'\gamma_\mu u_1}\,\underline{\bar u_2'\gamma^\mu u_2}\,\underline{\bar u_2\gamma^\nu u_1'}\,\underline{\bar u_1\gamma_\nu u_2'}\right\}$$

Lets explicitly put in the component indices:

$$\sum_{s_1}\sum_{s_2}\sum_{s_1'}\sum_{s_2'}\underline{\bar u'_{1\alpha}\gamma_{\mu,\alpha\beta}u_{1\beta}}\,\underline{\bar u'_{2\sigma}\gamma^\mu_{\sigma\lambda}u_{2\lambda}}\,\underline{\bar u_{2\theta}\gamma^\nu_{\theta\epsilon}u'_{1\epsilon}}\,\underline{\bar u_{1\delta}\gamma_{\nu,\delta\rho}u'_{2\rho}}$$

$$\sum_{s_2'}\sum_{s_1'}\sum_{s_1}\bar u'_{1\alpha}\gamma_{\mu,\alpha\beta}u_{1\beta}\bar u'_{2\sigma}\gamma^\mu_{\sigma\lambda}\Big(\underbrace{\sum_{s_2}u_{2\lambda}\bar u_{2\theta}}_{\left(\frac{\not{p}_2+m}{2m}\right)_{\lambda\theta}}\Big)\gamma^\nu_{\theta\epsilon}u'_{1\epsilon}\bar u_{1\delta}\gamma_{\nu,\delta\rho}u'_{2\rho}$$

$$\sum_{s_2'}\sum_{s_1'}\sum_{s_1}\bar u'_{1\alpha}\gamma_{\mu,\alpha\beta}u_{1\beta}\bar u'_{2\sigma}\left[\gamma^\mu\frac{\not{p}_2+m}{2m}\gamma^\nu\right]_{\sigma\epsilon}u'_{1\epsilon}\,\underline{u_{1\delta}\gamma_{\nu,\delta\rho}u'_{2\rho}}$$

$$\sum_{s_2'}\sum_{s_1'}\bar u'_{1\alpha}\gamma_{\mu,\alpha\beta}\underbrace{\sum_{s_1}u_{1\beta}u_{1\delta}}_{\left(\frac{\not{p}_1+m}{2m}\right)_{\beta\delta}}\gamma_{\nu,\delta\rho}u'_{2\rho}\bar u'_{2\sigma}\left[\gamma^\mu\frac{\not{p}_2+m}{2m}\gamma^\nu\right]_{\sigma\epsilon}u'_{1\epsilon}$$

$$\sum_{s_1'}\bar u'_{1\alpha}\gamma_{\mu,\alpha\beta}\left(\frac{\not{p}_1+m}{2m}\right)_{\beta\delta}\gamma_{\nu,\delta\rho}\sum_{s_2'}\underbrace{u'_{2\rho}\bar u_{2\sigma}}_{\left(\frac{\not{p}_2'+m}{2m}\right)}\left[\gamma^\mu\frac{(\not{p}_2+m)}{2m}\gamma^\nu\right]_{\sigma\epsilon}u'_{1\epsilon}$$

$$\sum_{s_1'}\bar u_{1\alpha}\left[\gamma_\mu\frac{\not{p}_1+m}{2m}\gamma_\nu\right]_{\alpha\rho}\left[\frac{\not{p}_2'+m}{2m}\right]_{\rho\sigma}\times\left[\gamma^\mu\frac{\not{p}_2+m}{2m}\gamma^\nu\right]_{\sigma\epsilon}u'_{1\epsilon}$$

$$\sum_{s_1'}\underbrace{u'_{1\epsilon}\bar u'_{1\alpha}}_{\left(\frac{\not{p}_1'+m}{2m}\right)_{\epsilon\alpha}}\left[\gamma_\mu\frac{\not{p}_1+m}{2m}\gamma_\nu\right]_{\alpha\rho}\left[\frac{\not{p}_2'+m}{2m}\right]_{\rho\sigma}\left[\gamma^\mu\frac{\not{p}_2+m}{2m}\gamma^\nu\right]_{\sigma\epsilon}$$

$$\sum_{e,\alpha} \left(\frac{\not{p}_1'+m}{2m}\right)_{\epsilon\alpha} \left(\gamma_\mu \cdot \frac{\not{p}_1+m}{2m}\gamma_\nu \cdot \frac{\not{p}_2'+m}{2m}\ \gamma^\mu \frac{\not{p}_2+m}{2m}\gamma^\nu\right)_{\alpha\epsilon}$$

↳ I've reintroduced explicitly the summation over spinor indices...

$$\sum_{e} \left\{\left(\frac{\not{p}_1'+m}{2m}\right)\left(\gamma_\mu \frac{[\not{p}_1+m]}{2m}\gamma_\nu\right)\left(\frac{\not{p}_2'+m}{2m}\right)\left(\gamma^\mu \frac{[\not{p}_2+m]}{2m}\gamma^\nu\right)\right\}_{\epsilon\epsilon}$$

$$= Tr\left\{\left(\frac{\not{p}_1'+m}{2m}\right)\left(\gamma_\mu \frac{(\not{p}_1+m)}{2m}\gamma_\nu\right)\left(\frac{\not{p}_2'+m}{2m}\right)\left(\gamma^\mu \frac{(\not{p}_2+m)}{2m}\gamma^\nu\right)\right\}$$

↳ (6)

It is straightforward to work out the second additive term in (5) it is in fact the same.

Now we have to take traces etc. It is often the case that the electrons used are highly relativistic, $E \gg m$. In this limit we'll have:

(Below:)

$$Tr(\not{p}_1'\gamma_\mu \not{p}_1 \gamma_\nu \not{p}_2'\gamma^\mu \not{p}_2 \gamma^\nu) \times \frac{1}{16m^4}$$

This is the trace of a product of <u>eight</u> γ matrices.

Before I can tackle this, I'll have to proove a couple more identities involving gamma matrices.

Theorem <u>IV</u>

(a) $\gamma_\mu\gamma^\mu = 4\mathbb{1}$ where $\mathbb{1}$ is the identity matrix.

$$\gamma_\mu\gamma^\mu = \gamma_0\gamma^0 + \gamma_1\gamma^1 + \gamma_2\gamma^2 + \gamma_3\gamma^3$$

recall $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} = 2g^{\mu\nu} = \{\gamma^\mu, \gamma^\nu\}$

if $\mu = \nu$ $\gamma^\mu\gamma^\mu = g^{\mu\mu} = 1$ for $\mu = 0$, -1 for $\mu = 1, 2, 3$

$\gamma_\mu\gamma^\mu = \gamma_0^0\gamma^0 - \gamma^1\gamma^1 - \gamma^2\gamma^2 \gamma^3\gamma^3 = (1+1+1+1)\mathbb{1}$
$= 4\mathbb{1}$.

(b) $\gamma_\mu \not{a} \gamma^\mu = -2\not{a}$

$\gamma_\mu \not{a} \gamma^\mu = \gamma_\mu\gamma^\nu a_\nu \gamma^\mu = a_\nu\gamma_\mu(2g^{\mu\nu} - \gamma^\mu\gamma^\nu)$
$= a_\nu\gamma_\mu 2g^{\mu\nu} - a_\nu\gamma_\mu\gamma^\mu\gamma^\nu = 2a_\nu\gamma^\nu - 4a_\nu\gamma^\nu = -2\not{a}$

(c) $\gamma_\mu \not a \not b \gamma^\mu = 4a\cdot b\, \mathbb{1}$

$\Rightarrow \gamma_\mu \not a \not b \gamma^\mu = \gamma_\mu \not a \gamma^\lambda b_\lambda \gamma^\mu = \gamma_\mu \not a\, b_\lambda \gamma^\lambda \gamma^\mu$

$= \gamma_\mu \not a\, b_\lambda [2g^{\lambda\mu} - \gamma^\mu \gamma^\lambda] = \gamma_\mu \not a\, 2g^{\lambda\mu} b_\lambda - \gamma_\mu \not a \gamma^\mu \gamma^\lambda b_\lambda$

$= \not b \not a \cdot 2 + 2\not a \not b = 2 a_\alpha b_\beta [2g^{\alpha\beta}] = 4a\cdot b\, \mathbb{1}.$

($\gamma_\mu \not a \gamma^\mu = -2\not a$ part (b)).

(d) $\gamma_\mu \not a \not b \not c \gamma^\mu = -2\not c \not b \not a$

let $\not c = \gamma^r c_r \Rightarrow \gamma_\mu \not a \not b \gamma^r c_r \gamma^\mu$

$= \gamma_\mu \not a \not b \gamma^r \gamma^\mu c_r = \gamma_\mu \not a \not b (2g^{r\mu} - \gamma^\mu \gamma^r) c_r$

$= \gamma_\mu \not a \not b\, 2g^{r\mu} c_r - \gamma_\mu \not a \not b \gamma^\mu \gamma^r c_r = 2\not c \not a \not b - \underbrace{\gamma_\mu \not a \not b \gamma^\mu}_{4a\cdot b} \not c$

$\Rightarrow 2\not c \not a \not b - 4a\cdot b \not c$

$2\not c \gamma^\mu \gamma^r a_\mu b_r - 4a\cdot b \not c$

$2\not c [2g^{\mu r} - \gamma^r \gamma^\mu] a_\mu b_r - 4a\cdot b \not c$

$-2\not c \not b \not c + 4g^{\mu r} a_\mu b_r - 4a\cdot b \not c$

$-2\not c \not b \not a \;\; \cancel{4a\cdot b \not c} - \cancel{4a\cdot b \not c} = \gamma_\mu \not a \not b \not c \gamma^\mu$ QED.

Back to page 6 expression (6)

$Tr[\not p_1' \gamma_\mu \not p_1 \gamma_r \not p_2' \gamma^\mu \not p_2 \gamma^r] = Tr[\gamma^r \not p_1' \gamma_\mu \not p_1 \gamma_r \not p_2' \gamma^\mu \not p_2]$

$= Tr[\underbrace{\gamma^r \not p_1' \gamma_\mu \not p_1 \gamma_r} \not p_2' \gamma^\mu \not p_2]$

by Theorem IV part (d) this is simply $-2\not p_1 \gamma_\mu \not p_1'$

$\Rightarrow \frac{1}{16m^4} Tr[-2 \not p_1 \gamma_\mu \not p_1' \not p_2' \gamma^\mu \not p_2] = -2\, Tr[\not p_1 \underbrace{\gamma_\mu \not p_1' \not p_2' \gamma^\mu}_{} \not p_2] \times \frac{1}{16m^4}$

by theorem IV part (d) $= 4\, p_1'\cdot p_2'$

$-8(p_1'\cdot p_2')\, Tr[\not p_1 \not p_2] = -8[p_1'\cdot p_2'][4 p_1\cdot p_2]$

$\hookrightarrow$ from a theorem proved a long time ago.

$= -32(p_1'\cdot p_2')(p_1\cdot p_2) \times \frac{1}{16m^4}$

$= \frac{-2(p_1'\cdot p_2')(p_1\cdot p_2)}{m^4}$ $\quad \times \frac{1}{4}$ for the spin average

$\times 2$ because there are two such terms

$= \frac{-(p_1'\cdot p_2')(p_1\cdot p_2)}{m^4}$ $\quad \times(-1)e^4$ these were factored out earlier.

Collecting together all the terms, from the previous term and expressions (3) and (4) on page 4; remember we will set $(m)^n$ terms $= 0$ for $n \geq 1$ but <u>not</u> <u>in</u> <u>the</u> <u>denominator</u>! $(E \gg m)$.

$$\frac{e^4}{m^4}\frac{(4\pi)^2}{2}\left\{\frac{(P_1'\cdot P_2')(P_1\cdot P_2)+(P_2'\cdot P_1)(P_1'\cdot P_2)}{(P_1-P_1')^4}+\frac{2(P_1'\cdot P_2')(P_1\cdot P_2)}{(P_1-P_1')^2(P_1-P_2')^2}\right.$$
$$\left.+\frac{(P_2'\cdot P_1')(P_1\cdot P_2)+(P_2'\cdot P_2)(P_1\cdot P_1')}{(P_1-P_2')^4}\right\} = |M_{fi}|^2_{\text{spin averaged}}$$
$$\hookrightarrow (7)$$

Now that we've got the $|\text{matrix element}|^2$ and averaged over spins, we will have to perform the integration over angular variables.

Going back to expression (1) page 2:

$$d\sigma = \frac{m^2}{\sqrt{(P_1\cdot P_2)^2-(m^2)^2}}\,|M_{fi}|^2_{\text{spin}}\,(2\pi)^4\,\delta(P_2'+P_1'-P_2-P_1)\times \frac{m}{(2\pi)^3}\times\frac{m}{(2\pi)^3}\frac{d^3P_1'}{E_1'}\frac{d^3P_2'}{E_2'}$$

We will now begin to manipulate this. The customary set-up for such experiments is in the centre of mass frame; both beams of electrons are impingent on each other with momenta equal in magnitude but opposite in direction.
This means that if $P_1 = (E, \vec{p})$, then $P_2 = (E, -\vec{p})$; it also means that if $P_1' = (E', \vec{p}\,')$ then $P_2' = (E', -\vec{p}\,')$.
We will begin to use the above information step by step; the first thing we simplify is the flux factor:

$$\frac{m^2}{\sqrt{(P_1\cdot P_2)^2-(m^2)^2}} = \frac{m^2}{\sqrt{(E^2+p^2)^2-(m^2)^2}} \qquad p^2 = |\vec{p}|^2$$

$\hookrightarrow$ 3-momentum.

$$= \frac{m^2}{\sqrt{(E^2+p^2-m^2)(E^2+p^2+m^2)}} = \frac{m^2}{\sqrt{2p^2\,2E^2}} = \frac{m^2}{2pE}$$

$p^2 = |\vec{p}|^2$, $|\vec{p}| = p$
(I was using shorthand!).

Collecting the $\frac{1}{(2\pi)^3}$'s etc:

$$d\sigma = \frac{m^4}{2|\vec{p}|E} \frac{1}{(2\pi)^2} |M_{fi}|^2_{spin} \, \delta(p_2' + p_1' - p_2 - p_1) \frac{d^3p_1'}{E_1'} \frac{d^3p_2'}{E_2'}$$

Now lets integrate out $d^3p_2' \Rightarrow$ observe p_1'

note $\frac{d^3p_2'}{E_2'} = 2\frac{d^3p_2'}{2E_2'} = 2 \cdot \theta(p_{20}') \, \delta(p_2'^2 - m^2) \, d^4p_2'$

$$d\sigma = \frac{m^4}{2|\vec{p}|E} \underbrace{\frac{1 \times 2}{(2\pi)^2} \int}_{\text{two integrals } d^3p_1' + d^4p_2'} |M_{fi}|^2_{spin} \, \delta(p_2' + p_1' - p_2 - p_1) \, \theta(p_{20}') \, \delta(p_2'^2 - m^2) \times d^4p_2' \frac{d^3p_1'}{[E_1']}$$

integrate over the first δ function to obtain

$p_2' = p_2 + p_1 - p_1' = (2E - E', -\vec{p}_1')$ (check last paragraph on pg 8).

$$d\sigma = \frac{m^4}{|\vec{p}|E} \frac{1}{(2\pi)^2} \int |M_{fi}|^2_{spin} \, \theta(p_{20}') \, \delta((p_2 + p_1 - p_1')^2 - m^2) \frac{d^3p_1'}{(E_1')}$$

where an integral remains:

note $\theta(p_{20}') = \theta(p_{20} + p_{10} - p_{10}') = \theta(2E - E')$

a dodgy way of representing a 4-vector.

$$d\sigma = \frac{m^4}{|\vec{p}|E} \frac{1}{(2\pi)^2} \int |M_{fi}|^2_{spin} \, \theta(2E - E') \, \delta[(2E - E', -\vec{p}_1')^2 - m^2] \times \frac{|\vec{p}_1'|^2 d|\vec{p}_1'| \, d\Omega_1'}{[E_1']}$$

We won't be integrating over $d\Omega_1'$

$$\frac{d\sigma}{d\Omega_1'} = \frac{m^4}{|\vec{p}|E} \frac{1}{(2\pi)^2} \int |M_{fi}|^2_{spin} \, \theta(2E - E') \, \delta[(2E - E', -\vec{p}_1')^2 - m^2] \times \frac{|\vec{p}_1'|^2 d|\vec{p}_1'|}{[E_1']}$$

Since $E_1'^2 = |\vec{p}_1'|^2 + m^2$ & $dE_1' \times E_1' = d|\vec{p}_1'| \times |\vec{p}_1'|$ $\Rightarrow |\vec{p}_1'| d\vec{p}_1' = E_1' dE_1'$

$$\Rightarrow \frac{d\sigma}{d\Omega_1'} = \frac{m^4}{|\vec{p}|E (2\pi)^2} \int |M_{fi}|^2_{spin} \, \theta(2E - E') \, \delta[(2E - E', -\vec{p}_1')^2 - m^2] \times |\vec{p}_1'| \, dE_1'$$

$$\frac{|\vec{p}_1'|^2 d|\vec{p}_1'|}{|E_1'|} = dE_1' \times |\vec{p}_1'|$$

$\theta(2E - E')$ integrated over dE' simply means that E' is constrained to lie between 0 and $2E$ or it (and the integral) is zero.

$$\frac{d\sigma}{d\Omega_1'} = \frac{m^4}{|\vec{P}|E(2\pi)^2}\int_0^{2E} |M_{fi}|^2_{spin}\, \delta[4E^2 - 4EE_1' + \underbrace{E_1'^2 - |\vec{P}_1'|^2}_{m^2} - m^2]\, |\vec{P}_1'|\, dE_1'$$

$m^2 - m^2 = 0$

$$\frac{d\sigma}{d\Omega_1'} = \frac{m^4}{|\vec{P}|E(2\pi)^2}\int_0^{2E} |M_{fi}|^2_{spin}\, \delta[4E^2 - 4EE_1']\, |\vec{P}_1'|\, dE_1'$$

$f(E') = 4E^2 - 4EE'$

$\frac{\partial f(E')}{\partial E'} = -4E \qquad \left|\frac{\partial f(E')}{\partial E'}\right| = 4E.$

also the zero of the argument of the function needs to be calculated:

$4E^2 - 4EE_1' = 0 \qquad E_1' = E.$

$$\frac{d\sigma}{d\Omega_1'} = \frac{m^4}{|\vec{P}|E}\frac{1}{(2\pi)^2}|M_{fi}|^2_{spin}\frac{1}{(4E)}|\vec{P}_1'| \;—$$

(evaluated appropriately).

Now note if $E_1' = E$ then $E_2' = E$ as well

and $E_1'^2 = |\vec{P}_1'|^2 + m^2 = E^2 = |\vec{P}_1|^2 + m^2 = |\vec{P}|^2 + m^2$

$\therefore \quad |\vec{P}_1'| = |\vec{P}|$

$$\frac{d\sigma}{d\Omega_1'} = \frac{m^4}{4E^2}\frac{1}{(2\pi)^2} \times |M_{fi}|^2_{spin} \quad — (8)$$

Lets now recall that $|M_{fi}|^2_{spin}$ has to be written out explicitly.

page 8 expression (7)

$$\frac{e^4}{m^4}\frac{(4\pi)^2}{2}\left\{\frac{(P_1'\cdot P_2')(P_1\cdot P_2) + (P_2'\cdot P_1)(P_1'\cdot P_2)}{(P_1 - P_1')^4} + \frac{2(P_1'\cdot P_2')(P_1\cdot P_2)}{(P_1 - P_1')^2(P_1 - P_2')^2}\right.$$

$$\left. + \frac{(P_2'\cdot P_1')(P_1\cdot P_2) + (P_2'\cdot P_2)(P_1\cdot P_1')}{(P_1 - P_2')^4}\right\}$$

Before proceeding lets make absolutely clear that we understand the relationship between the incoming and outgoing four momenta ie: $P_1, P_2, P_1' P_2'$

(1). $P_1 = (E_1, \vec{P_1})$ $P_2 = (E_2, \vec{P_2})$ if this is the C-M (centre of mass) frame then $\vec{P_2} = -\vec{P_1}$ and $E_1 = E_2$ since $E_1 = \sqrt{|\vec{P_1}|^2 + m^2}$ and $E_2 = \sqrt{|\vec{P_2}^2| + m^2}$
So $P_1 + P_2$ can be written as: $(E, \vec{P}), (E, -\vec{P})$

(2). From 4-momentum conservation

$$P_1 + P_2 = P_1' + P_2' \Rightarrow (2E, 0) = (E_1' + E_2', \vec{P_1}' + \vec{P_2}')$$

$\therefore \vec{P_2}' = -\vec{P_1}'$ and $\therefore E_1' = \sqrt{|\vec{P_1}'|^2 + m^2} = E_2' = \sqrt{|\vec{P_1}'|^2 + m^2}$

$\therefore E_1' = E_2' = E'$ $\therefore$ $2E' = 2E$ $\therefore$ $E' = E$.

(3). So $\vec{p}_1' = -\vec{p}_2'$. So $P_1' = (E, +\vec{P}')$ $P_2' = (E, -\vec{P}')$
but $= E = \sqrt{|\vec{P}'|^2 + m^2}$ and also $\sqrt{|\vec{P}|^2 + m^2}$ $\therefore$
$|\vec{P}| = |\vec{P}'|$

So all these 4-momenta have the <u>same energy</u>, and the same magnitude of the 3-momentum. Although the momenta of the incoming and outgoing particles are <u>back to back</u>

$P_1 = (E, \vec{P})$
$P_2 = (E, -\vec{P})$
$P_1' = (E, \vec{P}')$ with $|\vec{P}'| = |\vec{P}|$
$P_2' = (E, -\vec{P}')$

Once again, we've merrily set $m = 0$ since $E \gg m$ $\therefore$ $E \simeq |\vec{P}| = |\vec{P}'|$ in this ultra-relativistic limit. Also denote the angle between $\vec{P}$ & $\vec{P}'$ by θ.

Lets begin to evaluate $\overline{|M_{fi}|^2}_{spin}$
Lets begin with q^2 & q^4s in the denominator if you think about it these are simply the momentum transfer terms.

$$(p_1 - p_1')^2 = p_1^2 + p_1'^2 - 2p_1\cdot p_1' = 2m^2 \xrightarrow{} 0 - 2[E^2 - \vec{p}\cdot\vec{p}\,']$$
$$-2[E^2 - |\vec{p}||\vec{p}\,'|\cos\theta] \simeq -2E^2 - 2E^2\cos\theta$$

Now $\cos\theta = 1 - 2\sin^2\theta/2$

$$\Rightarrow \quad -2E^2 + 2E^2 - 4\sin^2\theta/2 = 4\sin^2\theta/2$$

$$\therefore \; (p_1 - p_1')^4 = 16E^4\sin^4\theta/2$$

$\rightarrow \; p_2 = -\vec{p}\,'$

$$(p_1 - p_2')^2 = 2m^2 - 2[E^2 + \vec{p}\cdot\vec{p}\,']$$
$$= -2E^2 + (-2)E^2\cos\theta = -2E^2[1 + \cos\theta]$$
$$= -2E^2[1 + 2\cos^2\theta/2 - 1] = -4E^2\cos^2\theta/2$$

$$\therefore \; (p_1 - p_2')^4 = 16E^4\cos^2\theta/2$$

also $(p_1 - p_1')^2(p_1 - p_2')^2 = 16E^4\cos^2\theta/2\,\sin^2\theta/2$

So that was every denominator in $\overline{|M_{fi}|^2}_{spin}$ lets do the numerators.

$$p_1'\cdot p_2' = E^2 - \vec{p}_1\,'\cdot\vec{p}_2\,' = E^2 + |\vec{p}\,'|^2 \simeq 2E^2$$
$$p_1\cdot p_2 = E^2 - \vec{p}_1\cdot\vec{p}_2 = E^2 + |\vec{p}|^2 \simeq 2E^2$$
$$p_1\cdot p_2' = E^2 - \vec{p}\cdot(-\vec{p}\,') = E^2 + |\vec{p}|^2\cos^2\theta = 2E^2\cos^2\theta/2$$
$$p_1'\cdot p_2 = 2E^2\cos^2\theta/2$$
$$p_1\cdot p_1' = 2E^2\sin^2\theta/2$$

Invoking expression (8) page 10.

$$\frac{d\sigma}{d\Omega_1'} = \frac{\cancel{m^4}}{4E^2}\frac{1}{(2\pi)^2}\times\frac{e^4}{\cancel{m^4}}\times\frac{(4\pi)^2}{2}\left[\frac{4E^4 + 4E^4\cos^4\theta/2}{16E^4\sin^4\theta/2}\right.$$
$$\left. + \frac{4E^4 + 4E^4\sin^4\theta/2}{16E^4\cos^4\theta/2} + \frac{2\times(2E^2)(2E^2)}{16E^4\sin^2\theta/2\cos^2\theta/2}\right]$$

$$\frac{d\sigma}{d\Omega_1'} = \frac{1}{4E^2\times 4\pi^2\times 2}16\pi^2\times e^4\left[\frac{1+\cos^4\theta/2}{4\sin^4\theta/2} + \frac{1+\sin^4\theta/2}{4\cos^4\theta/2} + \frac{2}{4\sin^2\theta/2\cos^2\theta/2}\right]$$

$$\frac{d\sigma}{d\Omega_1'} = \frac{e^4}{8E^2}\left[\frac{1+\cos^4\theta/2}{\sin^4\theta/2} + \frac{1+\sin^4\theta/2}{\cos^4\theta/2} + \frac{2}{\sin^2\theta/2\cos^2\theta/2}\right]$$

(I can't believe I got it right!) ☺