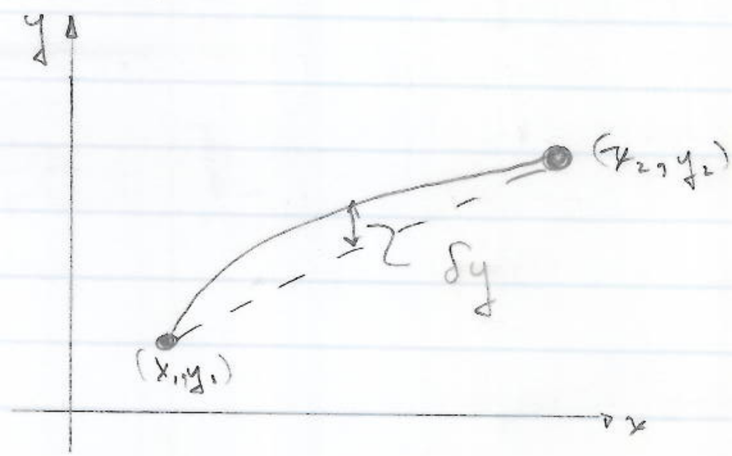
# Calculus of variations

Let $$I = \int_{x_1}^{x_2} f(y, y', x)\, dx$$

Under the integral sign $f$ is a <u>known</u> function of the indicated variables $y$, $y'$ and $x$, but the dependence of $y$ on $x$ IS NOT FIXED, that is $y = y(x)$ IS UNKNOWN. That is to say that although the integral is from $x_1$ to $x_2$, the exact path of integration IS NOT KNOWN




We are to choose the path of integration through points $(x_1, y_1)$ and $(x_2, y_2)$ to minimize $I$. Strictly speaking, we shall determine extreme values of $I$: minima, maxima or saddle points. In most cases of physical interest, however, the extreme will be a minimum.

We shall assume there exists an optimum path, that is, an acceptable path for which $I$ is an extremum. We will then compare $I$ for our (unknown) optimum path with that obtained from neighboring paths. In the figure above, two possible paths are shown (there are an infinite number of possibilities, of course). The difference between these two paths for a given $x$ is called the variation of $y$, <u>$\delta y$</u>, and is conveniently described by introducing a new function <u>$\eta(x)$</u> to define the arbitrary deformation of the path and a scale factor <u>$\epsilon$</u> to give the magnitude of the variation

The function $\eta(x)$ is arbitrary except for two restrictions

① $\eta(x_1) = \eta(x_2) = 0$

which means that the function must pass through the fixed end points

② The function $\eta(x)$ must be differentiable.

Then, with the path described with $\epsilon$ and $\eta(x)$

$$Y(x) = y(x) + \epsilon\eta(x)$$

and

$$\delta y = Y(x) - y(x) = \epsilon\eta(x)$$

Let us choose $y(x)$ as the unknown path that will minimize $I$. Then $Y(x)$ describes a neighboring path.

In the figure above, $I$ is now a function of the parameter $\epsilon$.

$$I(\epsilon) = \int_{x_1}^{x_2} f(Y, Y', x)\, dx$$

And our condition for an extreme value is that

$$\left.\frac{\partial I}{\partial \epsilon}\right|_{\epsilon=0} = 0$$

⇒ This is analogous the vanishing of the derivative $dy/dx$ in differential calculus.

$$\frac{\partial I}{\partial \epsilon} = \int_{x_1}^{x_2} \left[ \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial \epsilon} + \frac{\partial f}{\partial Y'}\frac{\partial Y'}{\partial \epsilon} \right] dx$$

$$Y = y(x) + \epsilon\eta(x) \quad \text{and} \quad Y' = y'(x) + \epsilon\eta'(x)$$

Now $\quad \dfrac{\partial Y}{\partial \epsilon} = \eta(x) \quad$ and $\quad \dfrac{\partial Y'}{\partial \epsilon} = \eta'(x)$

$$\frac{\partial I}{\partial \epsilon}\bigg|_{\epsilon=0} = \int_{x_1}^{x_2} \left( \frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\eta'(x) \right) dx$$

Integrating the second term by parts, we obtain

$$\int_{x_1}^{x_2} \eta' \frac{\partial f}{\partial y'}\, dx = \underbrace{\eta(x)\frac{\partial f}{\partial y'}\bigg|_{x_1}^{x_2}}_{=0 \text{ because } \eta(x_1)=\eta(x_2)=0} - \int \eta(x) \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx$$

and

$$\frac{\partial I}{\partial \epsilon}\bigg|_{\epsilon=0} = \int_{x_1}^{x_2} \left[ \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right] \eta(x)\, dx = 0$$

Since $\eta(x)$ is arbitrary, the existence of an extreme can only be satisfied if the bracketed term itself is identically equal to zero

$$(*) \quad \boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0} \leftarrow \text{Euler Equation.}$$

It is important to watch the meaning of $\frac{\partial}{\partial x}$ and $\frac{d}{dx}$ closely. For example if

$$f = f[y(x), x]$$

then

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

An alternate form of Euler's equation

$$\boxed{\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right)} \leftarrow \text{prove this.}$$

Proof

$$\frac{df}{dx} = \frac{d}{dx} f(y, y'; x) = \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} + \frac{\partial f}{\partial x} = y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + \frac{\partial f}{\partial x} \quad (1)$$

and

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = y''\frac{\partial f}{\partial y'} + y'\frac{d}{dx}\frac{\partial f}{\partial y'} \quad (2)$$

(1) → (2) gives

$$= \frac{df}{dx} - \frac{\partial f}{\partial x} \underbrace{- y'\frac{\partial f}{\partial y} + y'\frac{d}{dx}\frac{\partial f}{\partial y'}}_{y'\left[\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y}\right] \to 0 \text{ By the Euler Equation}}$$

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) - \frac{df}{dx} + \frac{\partial f}{\partial x} = 0$$

In problems in which $f = f(y, y')$ and $x$ does not appear explicitly we have

$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$$

or

$$f - y'\frac{\partial f}{\partial y'} = \text{constant}$$

## Applications of the Euler equation

What is the shortest distance between two points in the x-y plane?

$$ds = [(dx)^2 + (dy)^2]^{1/2} = [1 + (y')^2]^{1/2}\,dx$$

The distance $I$ may be written

$$I = \int_{x_1 y_1}^{x_2 y_2} ds = \int_{x_1}^{x_2} [1 + (y')^2]^{1/2}\,dx$$

$$f(y, y', x) = [1 + y'^2]^{1/2}$$ ← x does not appear explicitly

$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$$

$$y' \frac{\partial f}{\partial y'} = \frac{y'^2}{[1+y'^2]^{1/2}}$$

$$f - y' \frac{\partial f}{\partial y'} = \frac{1}{[1+y'^2]^{1/2}}$$

$$-\frac{d}{dx}\left[\frac{1}{(1+y'^2)^{1/2}}\right] = 0$$

or

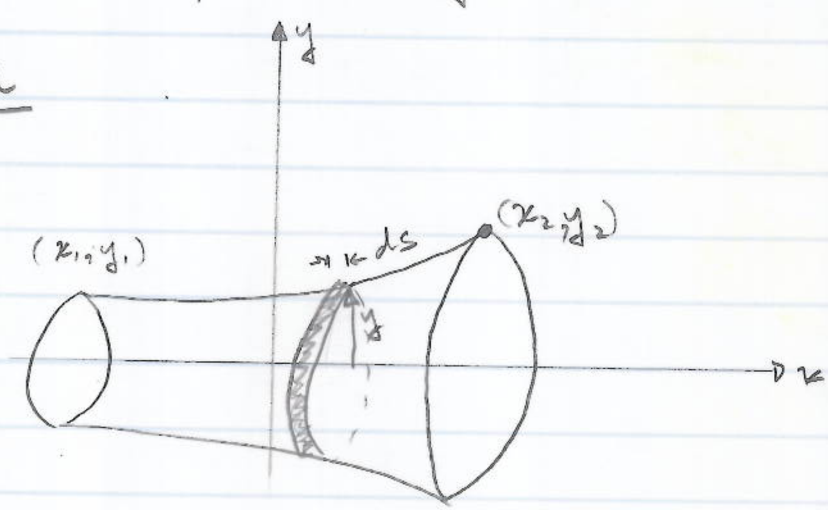
$$\frac{1}{(1+y'^2)^{1/2}} = \text{const}$$

This is satisfied by

$$y' = a \Rightarrow y(x) = ax + b$$

which is the equation for a straight line

Soap Film




Consider a surface of revolution generated by revolving a curve $y(x)$ about the $x$-axis

The variational problem is to choose the curve $y(x)$ so that the area of the resulting surface will be a minimum

For the element of area shown in the figure above

$$dA = 2\pi y\,ds = 2\pi y(1+y'^2)^{1/2}\,dx$$

The variational equation is then

$$I = \int_{x_1}^{x_2} 2\pi y(1+y'^2)^{1/2}\,dx$$

Neglecting the $2\pi$ factor

$$f(y, y', x) = y(1+y'^2)^{1/2}$$

Since $\frac{\partial f}{\partial x} = 0$ we may apply the equation

$$-\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$$

and get

$$y(1+y'^2)^{1/2} - \frac{yy'^2}{(1+y'^2)^{1/2}} = c_1$$

or

$$\frac{y}{(1+y'^2)^{1/2}} = c_1$$

Squaring, we get

$$\frac{y^2}{1+y'^2} = c_1^2$$

and

$$(y')^{-1} = \frac{dx}{dy} = \frac{c_1}{\sqrt{y^2 - c_1^2}} \qquad c_1^2 \leq y_{min}^2 .$$

This may be integrated to give

$$x = c_1 \cosh^{-1} \frac{y}{c_1} + c_2$$

Solving for y yields

$$y = c_1 \cosh\left(\frac{x - c_2}{c_1}\right)$$

and $c_1$ and $c_2$ are determined by requiring the hyperbolic cosine to pass through the points $(x_1, y_1)$ and $(x_2, y_2)$

Our minimum area of revolution is a
CATENARY of REVOLUTION
or a
CATENOID.

## Brachistochrone

The original problem that led to the development of the calculus of variations was the brachistochrone or shortest time problem

A particle is sliding freely along a curve from $(x_1 y_1)$ to the origin under the influence of gravity. The curve is chosen to minimize the time of descent. Here the time of descent is given by

$$I = \int_0^{x_1 y_1} \frac{ds}{v}$$

Since by the conservation of energy

$$\frac{1}{2} m v^2 = mg(y_1 - y)$$

$$v = \sqrt{2g(y_1 - y)}$$

and

$$I = \int_0^{x_1} \left[ \frac{1 + y'^2}{2g(y_1 - y)} \right]^{1/2} dx$$

Ignoring the constant $2g$, our function $f$ is now

$$f = \left[ \frac{1 + y'^2}{y_1 - y} \right]^{1/2}$$

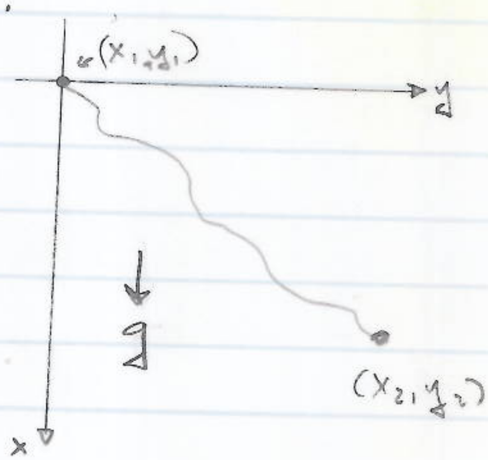
## Brachistochrone

$$t = \int_{x_1 y_1}^{x_2 y_2} \frac{ds}{v} = \int \frac{(dx^2+dy^2)^{1/2}}{\sqrt{2gx}} \qquad \frac{1}{2}mv^2 = mgx$$

$$= \int_{x_1=0}^{x_2} \left( \frac{1+y'^2}{2gx} \right)^{1/2} dx$$

We wish to minimize the time of transit. For the purposes of determining the variation of $t$, we need not consider the factor $(2g)^{-1/2}$.

Let
$$f = \left( \frac{1+y'^2}{x} \right)^{1/2}$$




From Euler's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left( \frac{\partial f}{\partial y'} \right) = 0$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow$$

$$\frac{\partial f}{\partial y'} = \text{const} \equiv (2a)^{-1/2}$$

$$\frac{\partial f}{\partial y'} = \frac{y'}{[x(1+y'^2)]^{1/2}} = (2a)^{-1/2}$$

$$\frac{y'^2}{x(1+y'^2)} = \frac{1}{2a}$$

$$\Rightarrow \quad y = \int \frac{x\,dx}{(2ax - x^2)^{1/2}}$$

Let

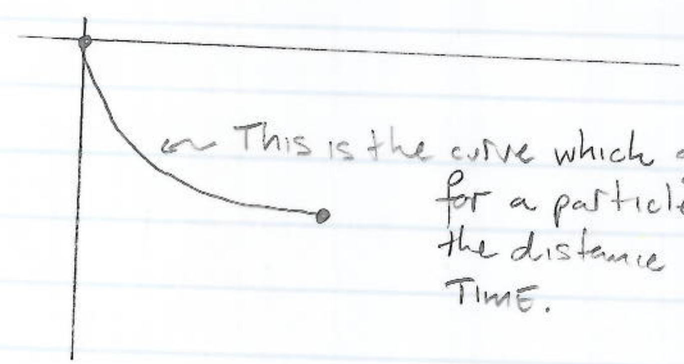
$$x = a(1-\cos\theta)$$
$$dx = a\sin\theta\, d\theta$$

$$y = \int a(1-\cos\theta)\, d\theta$$

$$= a(\theta - \sin\theta) + \text{constant}$$

And we have the parametric equations for a cycloid

$$\left.\begin{aligned} x &= a(1-\cos\theta) \\ y &= a(\theta - \sin\theta) \end{aligned}\right\}$$



This is the curve which gives the path for a particle that traverses the distance IN THE LEAST TIME.

We can design a cycloidal track and the particle will travel the distance in the least time.
[useful for roller coasters!]